Structural stability of attractor-repellor endomorphisms with singularities

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Abstract

We prove a theorem on structural stability of smooth attractor-repellor endomorphisms of compact manifolds, with singularities. By attractor-repellor, we mean that the non-wandering set of the dynamics f is the disjoint union of a repulsive compact subset with a hyperbolic attractor on which f acts bijectively. The statement of this result is both infinitesimal and dynamical. Up to our knowledge, this is the first in this hybrid direction. Our results generalize also a Mather's theorem in singularity theory which states that infinitesimal stability implies structural stability for composed mappings, to the larger category of laminations.

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1 Motivations and statement of the main result

1.1 Structural stability of Dynamical Systems

Let f be a *smooth endomorphism* of a compact manifold M. This means that f is a C^∞ map from M into itself which is not necessarily bijective and can have *singularities*: its tangent map can be non surjective at some points. For $k \in \mathbb{N} \cup \{\infty\}$, the map f is C^k*-structurally stable*, if for every f' close to f in the C^k-topology, there exists a homeomorphism h of M

such that the following diagram commutes:

$$\begin{array}{ccccc} & & f' & & \\ & M & \rightarrow & M & \\ h & \uparrow & & \uparrow & h \\ & M & \rightarrow & M & \\ & & f & & \end{array}$$

The topology of the space of C^k maps is the usual one for $k < \infty$, and for $k = \infty$ it is the union of the C^j topologies for $j \in \mathbb{N}$. The combined work of many authors (Smale [Sma67], Palis [PS70], De Melo [dM73], Robin [Rob71], Robinson [Rob76], Mañe [Mañ88]) has led to a satisfactory characterization of C^1-structurally stable diffeomorphisms. Let us state their characterization:

Theorem 1.1. *Let M be a compact connected manifold and f a C^1 diffeomorphism of M. Then the following conditions are equivalent:*

1. *f is C^1-structurally stable,*
2. *f satisfies Axiom A and the strong transversality condition,*
3. *f is C^0-infinitesimally stable.*

Let us recall the definitions of statements 2 and 3. A diffeomorphism f satisfies *Axiom A* if its non-wandering set Ω is hyperbolic and equal to the closure of the set of periodic points. The diffeomorphism f satisfies moreover the *strong transversality condition* if the stable and unstable manifolds of points of Ω intersect each other transversally.

The concept of infinitesimal stability will play a crucial role in this work. A C^{r+1} endomorphism is C^r-*infinitesimally stable* if the following map is surjective:

$$\sigma \in \chi^r(M) \mapsto \sigma \circ f - Tf \circ \sigma \in \chi^r(f),$$

with $\chi^r(M)$ the space of C^r sections of the tangent bundle TM, and with $\chi^r(f)$ the space of C^r sections of the push forward bundle f^*TM whose fiber at x is $T_{f(x)}M$.

In order to understand infinitesimal stability, let us regard a smooth endomorphism f of the torus $\mathbb{T} = \mathbb{R}^n/\mathbb{Z}^n$. We notice that the following map is Fréchet differentiable:

$$\phi \ : \ Diff^\infty(\mathbb{T}) \times Diff^r(\mathbb{T}) \to C^r(\mathbb{T}, \mathbb{T})$$

$$(f', h) \mapsto h \circ f' - f' \circ h,$$

where $Diff^\infty(\mathbb{T})$ and $Diff^r(\mathbb{T})$ are the spaces of C^∞ and C^r diffeomorphisms of $\mathbb{T}$ respectively. Moreover, its partial derivative at (f, id) with respect to the second variable is

$$\sigma \in \chi^r(\mathbb{T}) \mapsto \sigma \circ f - Tf \circ \sigma \in \chi^r(\mathbb{T}).$$

Consequently the above partial derivative is surjective iff f is C^r-infinitesimally stable.

1.2 Structural stability in Singularity Theory

Meanwhile, the school initiated by Whitney and Thom was interested in the following problem. Let f be a smooth map from a manifold M_1 into a manifold M_2. For $k > 0$, the map f is C^k-*equivalently stable*[1] if there exists a neighborhood U of f in the C^∞ topology such that for any $f' \in U$, there exist C^k automorphisms h_1 and h_2 of M_1 and M_2 respectively such that the following diagram commutes:

$$\begin{array}{ccccc} & & f' & & \\ & M_1 & \rightarrow & M_2 & \\ h_1 & \uparrow & & \uparrow & h_2 \\ & M_1 & \rightarrow & M_2 & \\ & & f & & \end{array} .$$

The problem is then to describe the C^k-equivalently stable maps which is usually simpler. This program was carried out by Mather who solved many conjectures of Thom. One of his results is the following:

Theorem 1.2 (Mather [Mat68a], [Mat69])**.** *The C^∞-equivalently stable, proper maps are the C^∞-equivalent infinitesimal stable maps.*

Here C^∞-*equivalent infinitesimal stability* means that the following map is onto:

$$(\sigma_1, \sigma_2) \in \chi^\infty(M_1) \times \chi^\infty(M_2) \mapsto \sigma_2 \circ f - Tf \circ \sigma_1 \in \chi^\infty(f).$$

1.3 Statement of the main result

Concerning the structural stability of endomorphisms, there is not yet a criterion, neither a satisfactory description. For instance, Dufour [Duf77] showed that there is no C^∞-infinitesimal stable endomorphism with a periodic point. On the other hand, the map $x \mapsto x^2$ on the one point compactification of $\mathbb{R}$, is C^0-infinitesimally stable but not C^∞-structurally stable.

There are very few old theorems stating sufficient conditions for the structural stability of endomorphisms that are not diffeomorphisms. Nowadays the subject regains of interest with new examples ([IPR08], [IPR97]). We recall below all these theorems. Our main theorem generalizes all the theorems implying the C^∞-stability of endomorphisms that are not diffeomorphisms.

A first old result was stated by Shub in his PhD thesis [Shu69], and requires the following definition:

Definition 1.3.1. Let f be a C^1 endomorphism of a Riemannian manifold (M, g), which sends a compact subset $K \subset M$ into itself. The compact subset K is *repulsive* if there

[1] Usually, in singularity theory, we say C^k-structurally stable; however this conflicts the dynamical terminology.

exist $\lambda > 1$ and $n \geq 1$ such that for every $x \in K$, the tangent map $T_x f^n$ is invertible with contractive inverse. The map f is *expanding* if M is compact and $K = M$.

Theorem 1.3 (Shub 69)**.** *The expanding C^1-endomorphisms of a compact manifold are C^1-structurally stable.*

The following result is known since many times:

Theorem 1.4. *Let f be a rational function of the Riemann sphere. If the Julia set of f is repulsive and if the critical points of f are quadratic, with non-preperiodic and disjoint orbits, then f is structurally stable for holomorphic perturbations.*

In the above example the singularities are not (equivalently) stable for smooth perturbations and so the dynamics is not C^∞-structurally stable. However, since its critical points are necessarily in the attracting basin of periodic attracting orbits, it is a good example of attractor-reppelor dynamics.

Definition 1.3.2. Let f be a smooth endomorphism of a compact, non necessarily connected manifold. The endomorphism f is *attractor-repellor* if its non-wandering set is the disjoint union of two compact subsets R and A such that:

- the compact subset R is repulsive, but non necessarily transitive,
- the compact subset A is hyperbolic, the restriction of f to A is bijective, and the unstable manifolds of points of A are contained in A. However A is not necessarily transitive.

From the above discussion we understand that our result on structural stability of endomorphisms needs to mix criteria from dynamical systems and singularity theory.

Theorem 1.5 (Main Result)**.** *Let f be an attractor-repellor, smooth endomorphism of a compact, non necessarily connected manifold M. If the following conditions are satisfied, then f is C^∞-structurally stable:*

(i) the periodic points of f are dense in A,

(ii) the singularities S of f have their orbits that do not intersect the non-wandering set Ω,

(iii) the restriction of f to $M \setminus \hat{\Omega}$ is C^∞-infinitesimally stable, with $\hat{\Omega} := cl\big(\cup_{n\geq 0} f^{-n}(\Omega)\big)$,

(iv) f is transverse to the stable manifold of A's points: for any $y \in A$, for any point z in a local stable manifold W^s_y of y, for any $n \geq 0$, and for any $x \in f^{-n}(\{z\})$, we have:

$$Tf^n(T_x M) + T_z W^s_y = T_z M.$$

We recall that the *singularities* of a smooth endomorphism f of M are the points x in M for which the tangent map $T_x f$ is not surjective.

Remark 1.3.3. Actually, we prove that the conjugacy map h between f and its perturbation is a smooth immersion restricted to each stable manifold of f without $\hat{A} := \cup_{n\geq 0} f^{-n}(A)$. Moreover, we prove that the partial derivatives along the stable manifolds depend continuously on the base point over all $W^s(\Omega) \setminus \hat{\Omega}$.

The following non-trivial remark follows easily from the proof of the main result:

Remark 1.3.4. The hypotheses of this theorem are open: any small smooth perturbation of f also satisfies them.

The main theorem generalizes a well known result:

Corollary 1.6. *Let f be a smooth endomorphism of the circle $\mathbb{S}^1$. If f is an attractor-repellor endomorphism such that the critical points of f are quadratic, with disjoint orbits in the basin of the attractor but non-preperiodic, then f is C^∞-structurally stable.*

Remark 1.3.5. Actually by [KSvS07], the above corollary is maximal in dimension 1: there are no more structurally stable endomorphisms of the circle. Moreover a C^∞-generic map of the circle satisfies this hypothesis.

Proof of Corollary 1.6. Only the infinitesimal condition hypothesis is not obvious. By Lemma 1.3.10 (see below), we only need to prove the C^∞-equivalent infinitesimal stability of the map $f:\ x \in \mathbb{R} \mapsto x^2 \in \mathbb{R}$. Let $\xi \in \chi(f)$. Let σ_2 be the constant function equal to $\xi(0)$ and:

$$\sigma_1(x) := \frac{\sigma_2(x^2) - \xi(x)}{2x} = -\int_0^1 \frac{\xi'(tx)}{2} dt$$

We notice that $\sigma_1, \sigma_2 \in \chi^\infty(\mathbb{S}^1)$ satisfy:

$$\sigma_2(x^2) - 2x\sigma_1(x) = \xi(x), \quad \forall x \in \mathbb{S}^1.$$

$$\text{i.e. } \sigma_2 \circ f - Tf \circ \sigma_1 = \xi.$$

□

Equivalent infinitesimal stability is a C^∞-generic property for maps from compact manifolds of dimension less than 9 (see [GG73] p163 when the differentiable map sends a manifold into one of different dimension). Conditions on the dimension of the manifold are given in [Nak89] when the singularities overlap.

This provides other applications of the main result.

Example 1.3.6. Let R be a rational function of the Riemann sphere such that all its critical points belong to the basin of attracting periodic orbits. Then the non-wandering set Ω of R split into two sets: the union of the basin of a finite number of attracting periodic points, and the Julia set J which is a repulsive compact subset. Thus R is an attractor-repellor endomorphism of the sphere. But condition (ii) is not satisfied since its singularities are

not stable. Nevertheless, for C^∞-generic perturbations R' of R the singularities Σ of R are equivalently stable and do not overlap along their orbits ($f^k(\Sigma)\cap\Sigma=\emptyset$, $\forall k>0$). Thus R' satisfies the hypothesis of the main theorem, and hence is structurally stable. To have the shape of the singularities of R' see [AGZV85] p. 20. This result was recently shown in [IPR97].

Example 1.3.7. When the endomorphism f does not have singularities, hypothesis (iii) is always satisfied, as we will see in Lemma 1.3.10. Consequently, our main result implies the structural stability of the following endomorphism:

$$f:\ \mathbb{S}^2\times\mathbb{S}^1\to\mathbb{S}^2\times\mathbb{S}^1$$

$$(z,z')\mapsto\Big(\frac{z+z'}{3},z'^2\Big),$$

where $\mathbb{S}^2$ is the Riemann sphere and $\mathbb{S}^1$ is the unit circle of the complex plane. We notice that the non-wandering set is the disjoint union of the Smale solenoid (inside the product of the unit disk with $\mathbb{S}^1$) with the repulsive circle $\{\infty\}\times\mathbb{S}^1$. This example is the first structurally stable endomorphism known which is neither a diffeomorphism nor expanding and was found in [IPR08].

Example 1.3.8. Let us introduce some singularities in the previous example. Let $\mathbb{D}$ be the closed unit disk. We notice that $\mathbb{D}\times\mathbb{S}^1$ contains the solenoid, is sent into its interior by the dynamics and the restriction of f to $\mathbb{D}\times\mathbb{S}^1$ is a diffeomorphism onto its image. Let us parametrize $\bar{\mathbb{T}}:=\mathbb{D}\times\mathbb{S}^1$ in polar coordinates $\{(re^{i\theta},z');\ r\in[0,1],\ \text{and}\ z'\in\mathbb{S}^1\}$. Let $\rho\in C^\infty([0,1],[0,1])$ be a Morse function equal to the identity everywhere except a small subset U close to 1 but with closure disjoint from $\{1\}$. We suppose that ρ has exactly two singularities and that ρ sends U into U. We notice that $\rho_1:\ re^{i\theta}\mapsto\rho(r)e^{i\theta}$ has for singularities two folds over two disjoint circles.

For some perturbation ρ_2 of ρ_1, the diffeomorphism ρ_2 is still equal to the identity on the complement of $U\times\mathbb{S}^1$, its singularities consist of two folds over two disjoint circles transverse to those of ρ_1, and ρ_2 sends $U\times\mathbb{S}^1$ into itself. Let us now regard:

$$f:\ (z,z')\mapsto\begin{cases} f(z,z') & \text{for } (z,z')\in(\mathbb{S}^2\times\mathbb{S}^1\setminus\bar{\mathbb{T}})\cup f^2(\bar{\mathbb{T}})\\ f(\rho_1(z),z') & \text{for } (z,z')\in\bar{\mathbb{T}}\setminus f(\bar{\mathbb{T}})\\ f^2\circ(\rho_2(\cdot,id)\circ f_{|\bar{\mathbb{T}}}^{-1}(z,z') & \text{for } (z,z')\in f(\bar{\mathbb{T}})\setminus f^2(\bar{\mathbb{T}})\end{cases}$$

Let us show how that the main theorem implies that f' is C^∞-structurally stable. First we notice that f' is a smooth endomorphism of $\mathbb{S}^2\times\mathbb{S}^1$, and is an attractor-repellor endomorphism: it has the same non-wandering set as f (i). Moreover f' is transverse to the stable manifolds of the solenoid (iv). Its singularities are formed by four folds; only two of them intersect the other ones along their orbits. This intersection occurs only at the first iteration

and is transverse. Since the singularity of ρ are close to $\{1\}$, the singularities of f' are disjoint from the solenoid, but included in $\bar{\mathbb{T}}$. As f embeds $\bar{\mathbb{T}}$ into itself, the orbits of the singularities do not intersect the non-wandering set (*ii*). Let us prove Property (*iii*). For this end we are going to show the C^∞-equivalent infinitesimal stability of the diagram:

$$\begin{array}{ccccc} & f_1 & & f_2 & \\ \mathbb{R}^2 & \to & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x^2,y) & & \\ & & (x,y) & \mapsto & \big((x+y)^2,y\big) \end{array}$$

The C^∞*-equivalent infinitesimal stability* of this diagram means that the map $(\sigma_1,\sigma_2,\sigma_3)\in \chi^\infty(\mathbb{R}^2)^3 \mapsto (\sigma_3\circ f_1 - Tf_2\circ\sigma_1, \sigma_3\circ f_2 - Tf_3\circ\sigma_1)\in\chi^\infty(f_1)\times\chi^\infty(f_2)$ is surjective. Such an infinitesimal stability will be sufficient for our purpose.

The product of above maps with the identity of $\mathbb{R}$ provides a local model of the singularities of f. The equivalent infinitesimal stability of such a product follows easily from the one of the above diagram. Also by using a partition of the unity, we get the infinitesimal stability of f restricted to a neighborhood of the singularities. Then Lemma 1.3.10 implies the infinitesimal stability Property (*iii*).

Let us prove the C^∞-equivalent infinitesimal stability of the above diagram. Let $\xi\in\chi^\infty(f_1)$ and $\zeta\in\chi^\infty(f_2)$. We want to construct $\sigma_1,\sigma_2,\sigma_3\in\chi^\infty(\mathbb{R}^2)$ such that:

$$\sigma_2\circ f_1 - Tf_1(\sigma_1)=\xi \quad\text{and}\quad \sigma_3\circ f_2 - Tf_2(\sigma_2)=\zeta.$$

By putting an exponent 1 or 2 to denote the first or second component respectively, the above system is equivalent to:

$$\begin{aligned} \sigma_2^1(x^2,y)-2x\sigma_1^1(x,y)&=\xi^1(x,y)\\ \sigma_2^2(x^2,y)-\sigma_1^2(x,y)&=\xi^2(x,y)\\ \sigma_3^1\big((x+y)^2,y\big)-2(x+y)\big(\sigma_2^1(x,y)+\sigma_2^2(x,y)\big)&=\zeta^1(x,y)\\ \sigma_3^2\big((x+y)^2,y\big)-\sigma_2^2(x,y)&=\zeta^2(x,y) \end{aligned}$$

Therefore it is sufficient to solve the following system with $X:=x+y$ and $Y:=y$:

$$\sigma_1^1(x,y)=\frac{\sigma_2^1(x^2,y)-\xi^1(x,y)}{2x} \tag{1}$$

$$\sigma_2^1(X,Y)+\sigma_2^2(X,Y)=\frac{\sigma_3^1\big(X^2,Y\big)-\zeta^1(X,Y)}{2X} \tag{2}$$

$$\sigma_1^2(x,y)=\sigma_2^2(x^2,y)-\zeta^2(x^2,y) \tag{3}$$

$$\sigma_2^2(X,Y)=\sigma_3^2\big(X^2,Y\big)-\zeta^2(X,Y) \tag{4}$$

We put $\sigma_3^1(X,Y):=\zeta(0,Y)$ which makes sense to (2) when X approaches 0 and

$$\sigma_3^2(Y^2,Y):=-\xi^1(Y,Y)+\frac{\zeta^1(0,Y)-\zeta^1(Y,Y)}{2Y}+\zeta^2(Y,Y)$$

and smoothly extended off the parabola $X = Y^2$.

Thus σ_2^2 is defined without ambiguity by (4), σ_2^1 by (2), and σ_1^2 by (3).

Finally we compute that $\sigma_2^1(0, y) = \xi^1(0, y)$ and so we can well define σ_1^1 by (1).

Example 1.3.9. Let M be the 2-sphere that we identify with the one-point compactification of $\mathbb{R}^2$. Let (r, θ) be the polar coordinates of $\mathbb{R}^2$.

$$\text{Let} \quad f : \hat{\mathbb{R}}^2 \to \hat{\mathbb{R}}^2$$
$$(r, \theta) \mapsto \Big(\frac{r}{2 + 2r^2}, \theta\Big).$$

We notice that f is an attractor-repellor endomorphism with $\{0\}$ as attractor and with empty repellor. Also the singularities of f are folds that do not overlap. Consequently the hypotheses of the main theorem are satisfied and so f is structurally stable.

Here is the lemma that we needed for all our computations.

Lemma 1.3.10. *Let f be an attractor-repellor endomorphism of a compact manifold M. Suppose that f satisfies Properties (i) and (ii) of the main theorem. Let Ω be the non-wandering set of f and $M' := M \setminus cl(\cup_{n \geq 0} f^{-n}(\Omega))$. Suppose moreover the existence of an open neighborhood $U \subset M'$ of the singularities such that:*

(a) for every $x \in U$, if an iterate $f^n(x)$ belongs to U, then $(f^k(x))_{k=0}^n$ belongs to U,

(b) The restriction of f^n to $U \setminus f^{-1}(U)$ is injective with injective derivative, for every $n \geq 0$,

(c) the map $\sigma \in \chi^\infty(M') \mapsto (\sigma \circ f - Tf \circ \sigma)_{|U} \in \chi^\infty(f_{|U})$ is surjective,

then the restriction of f to M' is C^∞-infinitesimally stable.

This lemma will be shown at this end of in Subsection 4.6.

1.4 Links with structural stability of composed of mappings

If we mentioned several times that the manifolds are not necessarily connected, it is because in this case, our main result is a complement to a theorem of Mather on structural stability of composed mapping. Moreover, contrarily to what happen for diffeomorphisms or local diffeomorphisms (such as expanding maps), the endomorphisms can send a connected component of the manifold into one of different dimension. As mentioned by Baas [Baa74], the problem of composed mapping was first stated by Thom, and have many applications in Biology (see [Tho71] and [Baa73]), in the study of network (see Baas [Baa74]) and in the study of the so-called Laudau singularity of Feynman integral (see [Pha67]).

To state the problem of composed mapping, let us consider a finite oriented graph $G := (V, A)$ with a manifold M_i associated to each vertex $i \in V$, and with a smooth map $f_{ij} \in C^\infty(M_i, M_j)$ associated to each arrow $[i, j] \in A$ from i to j.

Example 1.4.1.

$$\begin{array}{cccc} & & f_{12} & \\ & M_1 & \rightarrow M_2 & \\ f_{12} & \uparrow & \nearrow f_{32} & \\ & M_3 & & \\ & \circlearrowright & & \\ & f_{33} & & \end{array}$$

For $k \geq 0$, such a graph is C^k*-equivalently stable* if for every $(f'_{ij})_{[i,j]\in A}$ close to $(f_{ij})_{[i,j]\in A}$ in $\prod_{[i,j]\in A} C^\infty(M_i, M_j)$, there exist diffeomorphisms $(h_i)_i \in \prod_{i\in V} C^k(M_i, M_i)$ s.t. for every $[i,j] \in A$ the following diagram commutes:

$$\begin{array}{ccccc} & & f'_{ij} & & \\ & M_i & \rightarrow & M_j & \\ h_i & \uparrow & & \uparrow & h_j \\ & M_i & \rightarrow & M_j & \\ & & f_{ij} & & \end{array} .$$

The graph G is *convergent* if at most one arrow of A starts from each vertex. The graph is *without cycle* if for any family of arrows $([i_k, i_{k+1}])_{k=1}^N \in A^N$, the vertices i_1 and i_{N+1} are different. The following theorem was proved by Mather, and then written by Baas [Baa74]:

Theorem 1.7. *Let G be a graph of smooth proper maps, convergent and without cycle. The graph is C^∞-equivalently structurally stable if the following map is surjective:*

$$\prod_{i\in V} \chi^\infty(M_i) \to \prod_{[i,j]\in A} \chi^\infty(f_{ij})$$

$$(\sigma_i)_i \mapsto (Tf_{ij} \circ \sigma_i - \sigma_j \circ f_{ij})_{[ij]}.$$

We will see that our main theorem generalizes the above result when the manifolds $(M_i)_i$ are compact. Though this theorem has never been published, but it has been cited several times ([Nak89], [Duf77], [Buc77]...). Hence we will explain how to deduce the proof of this theorem in the non-compact case from this work (see Remark 4.2.8). But before, we shall notice that there is a canonical graph of maps associated to each smooth endomorphism f of a compact manifold M. Let $(M_i)_{i\in V}$ be the connected components of M. Let A be the set of arrows $[i,j]$ such that f sends M_i into M_j. Let f_{ij} be the restriction of f to M_i. Therefore, $G := (V, A)$ is a graph of smooth maps, convergent but *always with cycles*. Also, we notice that the C^∞-structural stability of f is the C^0-equivalent stability of this graph of maps.

Conversely, let us show that our main theorem implies the Mather's one on equivalent stability of graphs of maps, in the compact case. Let $G = (V, A)$ be a convergent and without cycle graph of maps of compact manifolds. Let $\hat{V}$ be the union of V with the circle $\hat{\mathbb{R}}$ and with the trivial 0-dimensional manifold $\{0\}$. We identify the circle $\hat{\mathbb{R}}$ to the one-point compactification of the real line. Let $f_{\hat{\mathbb{R}}\hat{\mathbb{R}}} := x \in \hat{\mathbb{R}} \mapsto 2x$ and let $f_{0\hat{\mathbb{R}}} := 0 \in \{0\} \mapsto 1 \in \hat{\mathbb{R}}$.

Let $V' \subset V$ be the subset of vertices from which no arrow starts. For $i \in V'$, let f_{i0} be the constant map from M_i onto $\{0\}$. Let finally $\hat{A} := A \cup \{[i,0]\}_{i\in V'} \cup \{[0,\hat{\mathbb{R}}]\} \cup \{[\hat{\mathbb{R}},\hat{\mathbb{R}}]\}$. One easily remarks that the hypotheses of Mather's theorem for the graph G implies those of the main result for the smooth endomorphism f of the disjoint union $M = \coprod_{i\in\hat{V}} M_i$, whose restriction to M_i is the map f_{ij}, with $i \in \hat{V}$ and $[i,j] \in \hat{A}$. Also for any smooth perturbation $(f'_{ij})_{[i,j]\in A}$ of $(f_{ij})_{[i,j]\in A}$, we can use the above algorithm to associate a perturbation f' of f. By the main theorem, the endomorphism f' is conjugated to f. By Remark 1.3.3, the conjugacy is smooth along the stable manifold of M. As they contain each manifold $(M_i)_{i\in V}$, this implies the Mather's theorem, in the compact case.

1.5 Thanks

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2 Proof of the main result

The proof is split into two parts. The first is the construction of a geometry which is preserved by the dynamics and which is persistent for small perturbations. For this end, we will use the formalism of laminations and some equivalent of the Hirsch-Pugh-Shub theory. This part is mostly dynamic and geometric. The second part consists of showing the structural stability of the dynamics with respects to this geometry. More precisely we will show a generalization of the Mather's theorem on equivalent stability of composing mappings in the larger context of laminations and by allowing these maps to overlap. This part uses mostly ingredients from singularity theory. As both parts use the formalism of laminations, we shall first define them rigorously.

2.1 Definition of the laminations and their morphisms

2.1.1 Definition of lamination

Let us consider a locally compact and second-countable metric space L covered by open subsets $(U_i)_i$ called *distinguished open subset*, endowed with homeomorphisms h_i from U_i onto $V_i \times T_i$, where V_i is an open set of $\mathbb{R}^d$ and T_i is a metric space. The *charts* $(U_i, h_i)_i$ define a (C^∞) *atlas* of a lamination structure on L if the *coordinate changes* $h_{ij} = h_j \circ h_i^{-1}$ can be written in the form:

$$h_{ij}(x,t) = (\phi_{ij}(x,t), \psi_{ij}(x,t)),$$

where ϕ_{ij} takes its values in $\mathbb{R}^d$, $\psi_{ij}(\cdot,t)$ is locally constant for any t, and the partial derivatives $(\partial_x^s \phi_{ij})_{s=1}^\infty$ exist and are continuous on *all* the domain of ϕ_{ij}. A *lamination* is such a metric space L endowed with a maximal C^∞-atlas $\mathcal{L}$. A *plaque* is a subset of L of the form $h_i^{-1}(V_i^0 \times \{t\})$, for a chart h_i, $t \in T_i$ and a connected component V_i^0 of V_i.

The *leaves* of $\mathcal{L}$ are the smallest subsets of L which contain any plaques that intersect them. If V is an open subset of L, the set of the charts $(U,\phi) \in \mathcal{L}$ such that U is included in V, forms a lamination structure on V, which is denoted by $\mathcal{L}_{|V}$.

The reader might read [Ghy99] and [Ber08] for example of laminations.

2.1.2 Morphisms of laminations

For $r \in [\![0,\infty]\!]$, a *C^r-morphism (of laminations)* from $(L,\mathcal{L})$ to $(L',\mathcal{L}')$ is a continuous map f from L to L' such that, seen via charts h and h', it can be written in the form:

$$h' \circ f \circ h^{-1}(x,t) = (\phi(x,t), \psi(x,t)),$$

where $\psi(\cdot,t)$ is locally constant, ϕ takes its values in $\mathbb{R}^{d'}$ and has its r-first derivatives with respect to x that are continuous on *all* the domain of ϕ. When $r=\infty$, the morphism is *smooth*. If, moreover, the linear map $\partial_x\phi(x,t)$ is always one-to-one (resp. onto), we will say that f is an *immersion (of laminations)* (resp. *submersion*). An *isomorphism (of laminations)* is a bijective morphism of laminations whose inverse is also a morphism of laminations. An *embedding (of laminations)* is an immersion which is a homeomorphism onto its image. An *endomorphism of* $(L,\mathcal{L})$ is a morphism from $(L,\mathcal{L})$ into itself. We denote by $T\mathcal{L}$ the vector bundle over L whose fiber at $x\in L$, denoted by $T_x\mathcal{L}$, is the tangent space at x to its leaf. If f is morphism from $\mathcal{L}$ to $\mathcal{L}'$, we denote by Tf the bundle morphism from $T\mathcal{L}$ to $T\mathcal{L}'$ over f induced by the differential of f along the leaves of $\mathcal{L}$.

2.1.3 Equivalent Classes of morphisms and their topologies

We will say that two morphisms f and f' from a lamination $(L,\mathcal{L})$ into a lamination $(L',\mathcal{L}')$ are *equivalent* if they send each leaf of $\mathcal{L}$ into the same leaf of $\mathcal{L}'$. We will deal with two topologies on such an equivalent class. Let us describe a base of neighborhoods of some morphism f from $\mathcal{L}$ to $\mathcal{L}'$ in each of these topologies. For this end, let us fix a cover $(K_i)_{i\in\mathbb{N}}$ of L by compact subsets s.t.:

- the cover $(K_i)_i$ is locally finite: any point of L has a neighborhood which intersects finitely many subsets of this family,
- for every i, the compact subsets K_i and $f(K_i)$ are included in distinguished open subsets endowed with charts (h_i,U_i) and (h'_i,U'_i) respectively.

We define (ϕ_i,ψ_i) by $h'_i \circ f \circ h_i^{-1} = (\phi_i,\psi_i)$ on $h_i(K_i)$.

Definition 2.1.1. In the *Whitney C^r-topology*, with $r < \infty$, a base of neighborhoods of f in its equivalent class $Mor^r_f(\mathcal{L}, \mathcal{L}')$ is given by the following open subsets with $(\epsilon_i)_i$ going all over the set of families of positive real numbers:

$$\Omega := \Big\{ f' \in Mor^r_f(\mathcal{L}, \mathcal{L}') \ : \ \forall i \geq 0\ f'(K_i) \subset U'_i \text{ and with } \phi'_i \text{ defined by}$$

$$h'_i \circ f' \circ h_i^{-1} = (\phi'_i, \psi_i), \text{ we have } \max_{h_i(K_i)} \Big(\sum_{s=1}^{r} \|\partial_x^s \phi_i - \partial_x^s \phi'_i\| \Big) < \epsilon_i \Big\}.$$

The *Whitney C^∞-topology* is the union of all the Whitney C^r-topologies, for $r \geq 1$. The Whitney C^r-topology is denoted by $\mathcal{W}^r$.

Definition 2.1.2. In the *compact-open C^r-topology* with $r < \infty$, a base of neighborhoods of f in its equivalent class is given by the following open subsets with $\epsilon > 0$ and $i \geq 0$:

$$\Omega := \Big\{ f' \in Mor^r_f(\mathcal{L}, \mathcal{L}') \ : \ f'(K_i) \subset U'_i \text{ and with } \phi'_i \text{ defined by}$$

$$h'_i \circ f' \circ h_i^{-1} = (\phi'_i, \psi_i), \text{ we have } \max_{h_i(K_i)} \Big(\sum_{s=1}^{r} \|\partial_x^s \phi_i - \partial_x^s \phi'_i\| \Big) < \epsilon \Big\}.$$

The C^∞*-compact-open topology* is the union of all the C^r-compact-open topologies, for $r \geq 1$. The compact-open C^r-topology is denoted by CO^r.

We remark that when L is compact, both topologies are the same.

We notice that when the laminations $(L, \mathcal{L})$ and $(L', \mathcal{L}')$ are manifolds, both definitions are consistent with the usual ones.

2.2 Results of the geometric part

The following theorem describes the geometry that the dynamics preserves:

Theorem 2.1. *If f is a C^∞-endomorphism of a smooth manifold M satisfying the hypotheses of the main result, then the stable manifolds of the points of the attractor A form the leaves of a C^∞-lamination $(L^s, \mathcal{L}^s)$.*

Moreover the compact set $\hat{R} := cl\big(\cup_{n\geq 0} f^{-n}(R)\big)$ is canonically endowed with a lamination structure $\mathcal{R}$ whose leaves are backward images by f of points of R. Moreover the lamination $(\hat{R}, \mathcal{R})$ is r-normally expanded, for any $r \geq 1$. This means that there exist $C > 0$ and $\lambda > 1$ s.t. for every $x \in R$, for every $u \in T_x\mathcal{R}$, $v \in T_x\mathcal{R}^\perp$, $n \geq 0$:

$$\|p \circ T_x f^n(v)\| \geq C \cdot \lambda^n \max(1, \|T_x f^n(u)\|^r),$$

with p the orthogonal projection of $TM_{|\hat{R}}$ onto $T\mathcal{R}^\perp$. Also M is equal to the disjoint union of L^s with $\hat{R}$.

We will prove this theorem in Section 3.

The following theorem states the persistence of this geometry:

Theorem 2.2. *Let f be a smooth endomorphism satisfying the hypotheses of the main result. Let $(R, \mathcal{R})$ and $(L^s, \mathcal{L}^s)$ be the laminations provided by Theorem 2.1. Let L'^s be a precompact, open subset of L^s whose closure is sent into L^s by f.*

Then, for any endomorphism f' C^∞-close to f, there exists two smooth embeddings i_R and i_s of respectively $(\hat{R}, \mathcal{R})$ and $(L'^s, \mathcal{L}^s_{|L'^s})$ into M and two smooth endomorphisms f'_R and f'_s of these laminations such that:

- *f'_R and f'_s are equivalent and CO^∞-close to the restrictions of f to respectively $\hat{R}$ and L'^s.*
- *i_R and i_s are CO^∞-close to the canonical inclusions of respectively $(\hat{R}, \mathcal{R})$ and $(L'^s, \mathcal{L}^s_{|L'^s})$ into M,*
- *the following diagrams commute:*

$$\begin{array}{ccccc} & & f' & & \\ & M & \rightarrow & M & \\ i_R & \uparrow & & \uparrow & i_R \\ & \hat{R} & \rightarrow & \hat{R} & \\ & & f'_R & & \end{array} \qquad \begin{array}{ccccc} & & f' & & \\ & M & \rightarrow & M & \\ i_s & \uparrow & & \uparrow & i_s \\ & L'^s & \rightarrow & L'^s & \\ & & f'_s & & \end{array} .$$

This theorem will be shown in Section 3.

Remark 2.2.1. Even if we do not need this to show the main theorem, by using [Ber07], we can easily show that the embeddings i_R and i_S are the restrictions of a homeomorphism of M which respects smoothly all the laminations.

2.3 Result of the singularity theory part

The main remaining difficulty of the proof of the main theorem is to conjugate f'_s to $f_{|L'^s}$. Such a conjugation follows basically from techniques of singularity theory, and require the following definition to be stated.

Definition 2.3.1. A morphism f from a lamination $(L', \mathcal{L}')$ into $(L, \mathcal{L})$ is *transversally bijective* if for any point $x \in L'$, there exist charts (U', ϕ') and (U, ϕ) of respectively x and $f(x)$ s.t. :

- f sends U into U',
- $\phi^{-1} \circ f \circ \phi'$ has its transverse component which is bijective.

The following theorem generalizes the one of Mather on equivalent stability of composed mapping of compact manifolds.

Theorem 2.3. *Let $(L, \mathcal{L})$ be a C^∞-lamination. Let C be a compact subset of L. Let f be a smooth endomorphism of $(L, \mathcal{L})$ satisfying:*

- *f is transversally bijective and proper,*

- *there exists $N > 0$ such that for every $x \in C$, an iterate $f^n(x)$ does not belong to C, for some $n \leq N$,*

- *the following map is surjective:*

$$\chi^\infty(\mathcal{L}) \to \chi^\infty(f)$$

$$\sigma \mapsto \sigma \circ f - Tf \circ \sigma,$$

  with $\chi^\infty(\mathcal{L})$ the space of C^∞-sections of $T\mathcal{L}$ and $\chi^\infty(f^*)$ the space of C^∞-sections of $f^*T\mathcal{L} \to L$.

Then for every f' close to f for the compact-open C^∞-topology, there exists an isomorphism $I(f')$ of $(L, \mathcal{L})$ such that for every $x \in C$:

$$f' \circ I(f')(x) = I(f') \circ f(x)$$

This theorem will be proved in the last section of this work.

2.4 Proof of the main theorem

Proof of Theorem 1.5. Le f be a smooth endomorphism of a compact manifold M, satisfying the hypotheses of the main theorem. Let $(\hat{R}, \mathcal{R})$ and $(L^s, \mathcal{L}^s)$ be the laminations provided by Theorem 2.1. Let f' be C^∞-close to f and let i_R, i_s and f'_s be the maps provided by Theorem 2.2. We remind that f'_s is close the restriction of f to the lamination $(L^s, \mathcal{L}^s)$ for the compact-open C^∞-topology.

We put $L := L^s \setminus \cup_{n\geq 0} f^{-n}(A)$.

We will show in Section 4.4 the following lemma:

Lemma 2.4.1. *The morphism f satisfies the hypotheses of Theorem 2.3.*

The difficulty of this lemma is to pass from infinitesimal stability for manifold to infinitesimal stability for lamination. This lemma together with Theorem 2.3 provide the structural stability of f on any compact subset of L. Let us chose carefully this compact subset. As A is an attractor on which f acts bijectively, there exists a small open neighborhood U of A such that:

- the restriction of f to U is a diffeomorphism,

- f sends $cl(U)$ into U,

- A is the maximal invariant subset of U: $A = \cap_{n\geq 0} f^n(U)$.

Let $U^* := \cup_{n>0} f^{-n}_{|U^c}(f(U))$. For U sufficiently small, the open subsets U^* and U have their closures disjoint from the singularities S and the critical values $f(S)$ of f.

Let $N > 0$ be such that $f^{-N}(U)$ contains S. Let K be the compact subset:

$$K := cl\Big(f^{-N}(U) \setminus (U^* \cup U)\Big).$$

We notice that K satisfies the following conditions:

(1) K is a neighborhood of the singularities and their image included in L,

(2) for every $x \in K$, if some iterate $f^k(x)$ belongs to K then all the iterates $(f^j(x))_{j=0}^k$ belong to K.

In order to apply Theorem 2.3, let us associate a perturbation $f^\#$ of $f_{|L}$ such that the restrictions $f^\#_{|K}$ and $f'_{s|K}$ are equal. For we take a function $\rho \in Mor^\infty(\mathcal{L}, \mathbb{R})$ with compact support and equal to 1 on K. Then given an isomorphism ϕ from $T\mathcal{L}$ to a neighborhood of the diagonal of $\mathcal{L} \times \mathcal{L}$ which sends a neighborhood of the zero section to the diagonal, we put:

$$f^\# :\ x \in L \mapsto \phi\big(\rho(x)\phi^{-1}(f(x), f'(x)\big).$$

We can now apply Theorem 2.3 with $C := K$ and for $f_{|L}$. Let $I(f'^\#)$ be the provided isomorphism of $(L, \mathcal{L})$ associated to $f'^\#$.

We define now:

$$h_0 :\ x \in \hat{K} \cup \hat{R} \mapsto \begin{cases} i_R(x) & \text{if } x \in \hat{R} \\ i_s \circ I(f'^\#)(x) & \text{if } x \in K \end{cases}$$

We notice that h_0 respects the lamination $\mathcal{L}_s$ and that for $x \in K \cup \hat{R}$ we have:

$$f' \circ h_0(x) = h_0 \circ f(x).$$

Let M_0 be the union of the connected components of M which intersect the non-wandering set Ω of f. We denote by K_0 the intersection $K \cap M_0$, by $D^n := f^n(U) \setminus f^{n+1}(U)$ for $n > 0$ and by $D^n := f^n_{|M_0}(K_0) \setminus f^{n+1}_{|M_0}(K_0)$ for any $n < 0$. We notice that $A \cup \bigcup_{n>0} D^n$ is a neighborhood of A. Let $R_0 := M_0 \cap \hat{R}$. Let us show that M_0 is contained in the disjoint union

$$A \cup K \cup R_0 \cup U^* \cup \bigcup_{n \neq 0} D^n.$$

Let $x \in M_0$. As the limit set of x is included in $\Omega = A \cup R_0$, every large iterate $f^n(x)$ is either close to A or close to R_0.

If $f^n(x)$ is close to A, then x is included in $A \cup \cup_{n \neq 0} D^n \cup U^*$. If the orbit of x is never close to A, then $(f^n(x))_n$ belongs to a small neighborhood of R_0, for n sufficiently large. As R_0 is repulsive and $f_{|M_0}$-invariant ($f^{-1}(R_0) \cap M_0 = R_0$), R_0 is locally maximal:

$$R_0 = \cap_{n \geq 0} f^{-n}(V), \quad \text{for some neighborhood } V \text{of } R_0.$$

This implies that $f^n(x)$ belongs to R_0. This completes the proof of the following inclusion:

$$M_0 \subset A \cup R_0 \cup U^* \cup K \cup \bigcup_{n\neq 0} D^n. \tag{5}$$

We endow M with an adapted metric to the hyperbolicity of R_0 and A. By restricting U and taking N sufficiently large, we can suppose that f' is uniformly expending on $\cup_{n<0} D^n \cup R_0$ and uniformly contracting along the leaves of $\mathcal{L}^s_{|U}$. Thus we can suppose $\epsilon > 0$ small enough and then f' close enough to f such that:

- for every $x \in cl(\cup_{n<0} D^n \cup R^0)$, the restriction of f' to any ball $B(x,\epsilon)$ centered at x and with radius ϵ is an expanding diffeomorphism onto a neighborhood of $B(f(x),\epsilon)$,
- For every $x \in U$, the restriction of f' to $\mathcal{L}'_x$ is contracting with precompact image in $\mathcal{L}'_{f(x)}$, where $\mathcal{L}'_x$ and $\mathcal{L}'_{f(x)}$ are the image by i_s of respectively the union of ϵ-$\mathcal{L}^s$-plaque containing x and $f(x)$ respectively.

For $x \in A$, let $h_A(x)$ be the unique element of the intersection $\cap_{n\geq 0} f'^n\big(\mathcal{L}'_{f^{-n}_{|U}(x)}\big)$.

$$\text{Let } h_1 := x \in M_0 \setminus U^* \mapsto \begin{cases} h_0(x) & \text{if } x \in R_0 \cup K \\ f'^n \circ h_0 \circ f^{-n}_{|U} & \text{if } x \in D^n,\ n > 0 \\ f'^{-1}_{|B(x,\epsilon)} \circ \cdots \circ f'^{-1}_{|B(f^{n-1}(x),\epsilon)} \circ h_0 \circ f^n(x) & \text{if } x \in D^{-n}, n > 0 \\ h_A(x) & \text{if } x \in A \end{cases}$$

We notice that h_1 is well defined since the union (5) is disjoint. Also since f' and f are conjugated via h_0 on K, the map h_2 is continuous on $M_0 \setminus (A \cup R_0 \cup U^*)$.

As h_0 is close to the identity, by commutativity of the diagram and expansion of R_0, for every $x \in R_0$, the intersection:

$$\cap_{n>0} f'^{-1}_{|B(x,\epsilon)} \circ \cdots \circ f'^{-1}_{|B(f^{n-1}(x),\epsilon)}\big(B(f^n(x),\epsilon\big)$$

is exponentially decreasing to $\{h_0(x)\}$. Consequently, for every $x' \notin R_0 \cup U^*$ close to x and so in D^{-n} with n large, $h_1(x)$ belongs to $f'^{-1}_{|B(x,\epsilon)} \circ \cdots \circ f'^{-1}_{|B(f^{n-1}(x),\epsilon)}\big(B(f^n(x),\epsilon)\big)$ and so is close to $h_0(x)$. Thus h is continuous at R_0.

For $x \in A$, we recall that the intersection $\cap_{n\geq 0} f'^n(\mathcal{L}'^{\epsilon}_{f^{-n}_{|U}(x)})$ is exponentially decreasing to $\{h_1(x)\}$. Also for every x' close to x and so in D^n with n large, the point $h_1(x')$ belongs to the intersection $\cap_{k=0}^{n'} f'^k(\mathcal{L}'^{\epsilon}_{f^{-k}_{|U}(x')})$ whose diameter is small for $n' < n$ large. As $(\mathcal{L}'_{|f^{-k}_{|U}(x')})_{k=0}^{n'}$ and $(\mathcal{L}'_{|f^{-k}_{|U}(x)})_{k=0}^{n'}$ are close, $h_1(x')$ and $h_1(x)$ are close. This proves that h_1 is continuous at A.

Since U^* has its closure disjoint from the singularities of f, f is a local diffeomorphism on a neighborhood of $cl(U^*) \cap M_0$. Consequently, for ϵ-small enough and then f' close enough to f', the restriction $f'_{|B(x,\epsilon)}$ is a diffeomorphism onto its image, for every $x \in U^*$. As the

connected components of U^* accumulate on R_0, we can define similarly for f' close enough to f:

$$h_2 :\ x \in M_0 \mapsto \begin{cases} h_1(x) & \text{if } x \in M_0 \setminus U^* \\ f'^{-1}_{|B(x,\epsilon)} \circ \cdots \circ f'^{-1}_{|B(f^{k-1}(x),\epsilon)} \circ h_1 \circ f^k(x) & \text{if } x \in f^{-k}_{|U^c}(U) \setminus f^{-k+1}(U) \end{cases}$$

The continuity of h_2 on $M_0 \setminus R_0$ follows from the conjugacy of f' and f via h_1 on $M_0 \setminus U^*$. The continuity of h_2 at R_0 is proved similarly as we did for h_1.

By expansiveness of $f_{|A\cup R_0}$, h_1 is injective on $A \cup R^0$. Also by definition, $h_1$is injective on $M^0 \setminus (A \cup R_0 \cup U^*)$. By using the attraction-repulsion of $A - R_0$, we get that h_1 is injective on $M \setminus U^*$. By local inversion of $f_{|U^*}$, we have the injectivity of h_2.

Therefore h_2 is a continuous injective map from M_0 into itself, C^0-close to the identity. By compactness of M_0 this implies that h_0 is a homeomorphism of M_0.

Let us finally construct h. Let K' be a neighborhood of the singularities, included in the interior of K and satisfying properties (1) and (2) of K. Let $M_s := M \setminus (M_0 \cup K')$. Since the restriction of f to the M_s is a submersion, we can foliate M_s by the two following ways. For $x \in M_s$, let $m_x > 0$ be minimal such that $y := f^{m_x}(x)$ belongs to M_0. Let F_y be the submanifold equal to $\cup_{k>0} f^{-k}(\{y\}) \setminus (M_0 \cup K')$. Let F'_y be the submanifold equal to $\cup_{k>0} f'^{-k}(\{y\}) \setminus (M_0 \cup K')$. We notice that $(F_y)_y$ and $(F'_y)_y$ are the leaves of two foliations on M_s. We notice that the restriction of the exponential map associated to the metric of M:

$$\phi :\ TF'^{\perp}_y \to M$$

$$(x, u) \mapsto \exp_x(u)$$

is a diffeomorphism from a neighborhood of the zero section of $TF'^{\perp}_y \to F'_y$ onto a neighborhood V of F'_y. Let $\pi_y :\ V \to F'_y$ be the composition of ϕ^{-1} with the projection $TF'^{\perp}_y \to F'_y$. Let K'' be a compact neighborhood of K' in $int(K)$. We notice that the following map is well defined for f' close enough to f.

$$h'_2 :\ x \in M \setminus (M_0 \cup K'') \mapsto \pi_{h_2(y)}(x), \quad \text{if } x \in F_y.$$

We notice that h_0 and h'_2 sends every point $x \in K \setminus K''$ into $F'_{h_2(y)}$.

Thus we may patch h'_2 and h_2 to a map $h :\ M \to M$ satisfying that:

- h is equal to h_2 on M_0, to h_0 on K'' and to h'_2 on $M \setminus (K \cup M_0)$,
- h sends any points $x \in M \setminus K''$ to a points of $F'_{h_2(y)}$, with $y := f^{m_x}(x)$,
- The restriction of h to L is a smooth embedding of $\mathcal{L}$ into M,
- h is a homeomorphism onto its image C^0-close to the canonical inclusion.

We notice that h is a homeomorphism of M close to the identity. Moreover, it satisfies:

$$f' \circ h = h \circ f.$$

□

3 Partition of the manifold by invariant and persistent laminations

3.1 Construction of the invariant laminations

In this subsection we prove Theorem 2.1 which states the existence of a splitting of M into two laminations $(L^s, \mathcal{L}^s)$ and $(\hat{R}, \mathcal{R})$.

3.1.1 The disjoint union of L with $\hat{R}$ is M

By the work of Przytycki [Prz77], there exists a neighborhood V of R s.t. R is the maximal invariant of V:

$$R = \bigcap_{n \geq 0} f^{-n}(V).$$

As the limit set is included in the non-wandering set, the last equality implies that the complement of the basin of A is $\cup_{n\geq 0} f^{-n}(R)$. Thus the last union is closed and so equal to $\hat{R}$. To summary we have:

$$\hat{R} = \bigcup_{n\geq 0} f^{-n}(R) \quad \text{and} \quad M = \hat{R} \cup L.$$

3.1.2 Construction of $\mathcal{R}$

As the forward orbit of the critical set does not intersect R, the backward images of point of R form a partition of $\hat{R}$ by compact submanifolds. Moreover these submanifolds depend continuously on the point, and so are the leaves of a lamination $\mathcal{R}$ on $\hat{R}$.

Let M_0 be the union of the connected components of M which intersect Ω. We notice that the restriction $\mathcal{R}_{|M_0 \cap \hat{R}}$ is a 0-dimensional lamination: its leaves are points. Also since M is compact, there exists $N \geq 0$ s.t. $f^{-N}(M_0) = M$. Consequently to prove that $(\hat{R}, \mathcal{R})$ is a normally expanded lamination, we only need to prove that $\hat{R} \cap M'$ is repulsive; this is proved in [IPR08] Lemma 1.

3.1.3 Construction of $\mathcal{L}^s$

The existence of a laminar structure $\mathcal{L}^s$ on L^s is a consequence of the following proposition:

Proposition 3.1. *Let $r \in \{1, \dots, \infty\}$, f a C^r-endomorphism of a manifold M and K a hyperbolic compact subset of M. We suppose that the singularities of f have their orbits disjoint from K and that for every $y \in K$, for every $n \geq 0$, the map f^n is transverse to a local stable manifold of y.*

Then the union $W^s(K)$ of stable manifolds of points of K is the image of a C^r-lamination $(L, \mathcal{L})$ immersed injectively.

Moreover if every stable manifold does not accumulate on K, then $(L, \mathcal{L})$ is a C^r-embedded lamination.

Proof. We endow M with an adapted metric d_M to the hyperbolic compact subset K. For a small $\epsilon > 0$, the *local stable manifold of diameter* ϵ *of* $x \in K$ is the set of points whose (forward) orbit is ϵ-distant to the orbit of x. Let $W^s_\epsilon(K)$ be the union of stable manifolds of points in K of diameter ϵ. For ϵ small enough, the closure of $W^s_\epsilon(K)$ is sent by f into $W^s_\epsilon(K)$ and supports a canonical C^r-lamination structure $\mathcal{L}_0$. Let C be the subset $W^s_\epsilon(K) \setminus f^2\big(cl(W^s_\epsilon(K))\big)$. For $i > 0$, we denote by C_i the set $f^{-i}(C)$ and by C_0 the set $W^s_\epsilon(K)$. Consequently, the union $\cup_{n\geq 0} C_n$ is equal to $W^s(K)$. Moreover, for $k, l \geq 0$, if C_k intersects C_l then $|k-l| \leq 1$.

Let us now construct a metric on $W^s(K)$ such that $(C_n)_n$ is an open cover and such that the topology induced by this metric on C_n is the same as the one of M. For $(x,y) \in W^s(K)^2$, we denote by $d(x,y)$:

$$\inf \Big\{ \sum_{i=1}^{n-1} d_M(x_i, x_{i+1});\ n > 0,\ (x_i)_i \in W^s(K)^n, \text{ such that}$$
$$x_1 = x,\ x_n = y,\ \forall i \exists j : (x_i, x_{i+1}) \in C_j^2 \Big\}.$$

We remark that d is a distance with the announced properties. Let L be the set $W^s(K)$ endowed with this distance. We notice that if every stable manifold does not accumulate on K, then the topology on L induced by this metric and the metric of M are the same. In other words L is embedded.

For $i > 0$, we now construct on the open subset C_i a C^r-lamination structure $\mathcal{L}_i$. Let $(U_k, h_k)_k$ be an atlas of $\mathcal{L}_{0|C}$ of the form:

$$h_k :\ U_k \to \mathbb{R}^{d_k} \times T_k$$
$$x \mapsto \big(\phi_k(x), \psi_k(x)\big)$$

Let $U'_k := f^{-i}(U_k)$ and

$$\psi'_k :\ U'_k \to T_k$$
$$x \mapsto \psi_k \circ f^i(x)$$

By shrinking a slice U_k (and hence U'_k) and by using the transversality of f, there exists a neighborhood T'_k of any $t \in T_k$, such that $(\psi'^{-1}_k(t'))_{t' \in T'_k}$ is a family of manifolds that are all diffeormorphic to $M_t := \psi'^{-1}_k(t)$ by a diffeomophism that depends C^r-continuously on $t' \in T'_k$. Let us denote by $\phi'_k : U''_k \to M_t$ this continuous family of C^r diffeormorphisms, with $U''_k := \Psi'^{-1}_k(T'_k)$. Now let $(U_\alpha, h_\alpha)_{\alpha \in A}$ be a C^r-atlas of the manifold M_t. For each α, let U^α_k be the open subset of U''_k equal to $\phi'^{-1}_k(U_\alpha)$. We notice that the map:

$$h^\alpha_k :\ U^\alpha_k \to \mathbb{R}^d_\alpha \times T'_k$$
$$x \mapsto (h_\alpha \circ \phi'_k(x), \psi'_k(x))$$

is a chart of an atlas of lamination $\mathcal{L}_i$ on C_i, for $k \geq 0$, $t \in T_k$, and $\alpha \in A$.

Moreover, for i, j consecutive, the restriction of $\mathcal{L}_i$ and $\mathcal{L}_j$ to $C_i \cap C_j$ are equivalent. Thus, the structures $(\mathcal{L}_i)_{i \geq 0}$ span a C^r-lamination structure $\mathcal{L}$ on L. $\square$

3.2 Persistence of the laminations

In this section we prove Theorem 2.2.

The existence of i_R and f_R follows from the fact that $(\hat{R}, \mathcal{R})$ is a compact, r-normally expanded lamination for any $r \geq 1$. Thus by Theorem 0.1 of [Ber08], the lamination $(\hat{R}, \mathcal{R})$ is C^r-persistent. This means that for f' C^r-close to f, there exist a C^r-embedding i'_R of $(\hat{R}, \mathcal{R})$ into M, and a C^r-endomorphism f_R of $(R, \mathcal{R})$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f'"] & M_0 \\ \hat{R} \arrow[u, "i'_R"] \arrow[r, "f_R"'] & \hat{R} \arrow[u, "i'_R"'] \end{tikzcd} .$$

Moreover i'_R is C^r-close to the canonical inclusion and f_R is C^r-close to $f_{|\hat{R}}$.

By Theorem 2.1, the lamination $(\hat{R}, \mathcal{R})$ embedded by i_R is actually of class C^∞. Thus to smooth i'_R and f_R we consider a smooth tubular neighborhood (see Section 1.5 of [Ber08]). This is the data of:

- a smooth laminar structure $\mathcal{F}$ on $F := TM_{|\hat{R}}/T\mathcal{R}$ such that the leaves of $\mathcal{F}$ are the preimages by $\pi: \ F = TM_{|\hat{R}}/T\mathcal{R} \to L$ of the leaves of $\mathcal{L}$ and such that π is a smooth submersion,

- an immersion I from the restriction of $\mathcal{F}$ to a neighborhood of the zero section 0_F, such that
$$I \circ 0_F = id_{\hat{R}}$$

By compactness of $\hat{R}$, there exists ϵ such that the restriction of the ball $\mathcal{F}^\epsilon_x$ centered at $x \in \hat{R}$ and with radius ϵ in the leaf of 0_x in $\mathcal{F}$ is sent diffeomorphically by I onto an open subset of M. Let F^ϵ_x be the intersection of $\mathcal{F}^\epsilon_x$ with the fiber F_x of $F \to \hat{R}$ at x. For f' close to f, the image by I of F^ϵ_x intersects transversally at a unique point $i_R(x)$ the image by i'_R of a plaque $\mathcal{L}_x$ of x. We notice that $i_{R|\mathcal{L}_x}$ is the composition of i with the holonomy from the transverse section $i(\mathcal{L}_x)$ to $i_R(\mathcal{L}_x)$ along the foliation $(F^\epsilon_{x'})_{x' \in \mathcal{L}_x}$. As these submanifolds and foliations are smooth, $i_{R|\mathcal{L}_x}$ is smooth. As these foliations and manifolds depend continuously on x, i_R is a smooth morphism of $(\hat{R}, \mathcal{R})$ into M. As $i'_R(\mathcal{L}_x)$ is C^r-close to $i_R(\mathcal{L}_x)$, i_R is C^r-close to i. In particular i_R is an immersion.

Also by construction i_R is injective and so, by compactness of $\hat{R}$, i_R is an embedding.

Let finally $f'_R := x \in R \mapsto \pi \circ I^{-1}_{|\mathcal{F}^\epsilon_{f(x)}} \circ f' \circ i_R(x)$. The composition of a smooth morphisms f'_R is smooth. Also one easily notes that f_R is equivalent and C^r close to $f_{|R}$.

The persistence of $(L'^s, \mathcal{L}^s_{|L'^s})$ is showed similarly, by using this time Theorem 3.1 of [Ber08].

4 Infinitesimal stability implies stability on the non-wandering set

In this section, we prove Theorem 2.3. Throughout this section we denote by $(L, \mathcal{L})$ a C^∞-lamination, C a compact subset of L. Let f be a proper, smooth endomorphism of the lamination $(L, \mathcal{L})$ s.t. for some $N \geq 0$, $x \in C$ there exists $n \leq 0$ s.t. $f^n(x)$ does not belong to C.

4.1 Stability under deformations

We are going to prove that infinitesimal stability $(\mathcal{I})$ implies another property called *stability under deformations* $(\mathcal{D})$.

4.1.1 Condition $\mathcal{D}$

Definition 4.1.1. A *deformation of* f is a smooth endomorphism F of the product lamination $(L \times \mathbb{R}, \mathcal{L} \times \mathbb{R})$, of the form:

$$F \ : \ (x, t) \in L \times \mathbb{R} \mapsto (f_t(x), t) \in L \times \mathbb{R},$$

and such that $f_0 = f$. We remind that the leaves of the lamination $\mathcal{L} \times \mathbb{R}$ are the product of the leaves of the lamination $\mathcal{L}$ with $\mathbb{R}$.

Definition 4.1.2. Let B be a neighborhood of $0 \in \mathbb{R}$. A deformation $F = (f_t)_t$ of f is *trivial relatively to* $C \times B$ if there exists a deformation H of the identity of $(L, \mathcal{L})$:

$$H : \ (x, t) \in L \times \mathbb{R} \mapsto (h_t(x), t),$$

such that for all $(x, t) \in C \times B$:

$$h_t \circ f(x) = f_t \circ h_t(x).$$

The automorphism H is a *trivialization* of F (relatively to $C \times B$).

Definition 4.1.3 (Condition $\mathcal{D}$)**.** We say that f is *stable under deformations relatively to* C if for any bounded ball B centered at 0, for any deformation $F : \ (x, t) \mapsto (f_t(x), t)$ of f CO^∞-close enough to $F_0 := (x, t) \mapsto (f(x), t)$ is trivial relatively to $C \times B$.

Also the following proposition is obvious:

Proposition 4.1. *If f is stable under k-deformations relatively to C, then f is* structurally stable relatively to C*: for any f' CO^∞-sufficiently close to f there exists an isomorphism h of $(L, \mathcal{L})$ such that for any $x \in C$, we have:*

$$h \circ f(x) = f' \circ h(x)$$

4.1.2 Sufficiency of the implication $I \to D$

The main remaining difficulty is to prove the following theorem:

Theorem 4.2. *Let f be a proper, C^∞ endomorphism of a lamination $(L, \mathcal{L})$. Let C be a compact subset of L such that for some $N > 0$ the intersection $\cap_{n=0}^{N} f^{-n}(C)$ is empty. If f is transversally bijective and if:*

$\mathcal{I}$) *the following map is surjective:*

$$\chi^\infty(\mathcal{L}) \to \chi^\infty(f)$$

$$\sigma \mapsto \sigma \circ f - Tf \circ \sigma,$$

Then:

$\mathcal{D}$) *The morphism f is stable under deformations relatively to C.*

By the previous proposition, this theorem implies Theorem 2.3.

4.2 Condition $\mathcal{I} \Rightarrow$ condition $\mathcal{D}$

Let f be infinitesimally stable. We want to prove that f satisfies condition $\mathcal{D}$. Let $\chi^\infty(\mathcal{L}, \mathbb{R})^0$ be the space of smooth vector fields with $\mathbb{R}$ component equal to 0. Let $\frac{\partial}{\partial t}$ be the canonical unit vector field of the product $\mathcal{L} \times \mathbb{R}$ associated to $\mathbb{R}$.

4.2.1 Thom-Levin Theorem

The following theorem transforms the problem of the existence of trivialization H to a linear problem. This will allow us to solve this problem algebraically.

Theorem 4.3 (Thom-Levine theorem adapted)**.** *Let B be a subset of $\mathbb{R}$ and let W be a neighborhood of $C \times B$ in $L \times \mathbb{R}$. There exists a $\mathcal{W}^\infty$-neighborhood V_ξ of $0 \in \chi^\infty(\mathcal{L} \times \mathbb{R})^0$. A deformation F of f is trivial relatively to $C \times B$, if there exists $\xi \in V_\xi$ such that on W:*

$$\tau_F := TF \circ \frac{\partial}{\partial t}_{|L \times \mathbb{R}} - \frac{\partial}{\partial t} \circ F = TF \circ \xi - \xi \circ F.$$

Remark 4.2.1. Actually the statement of the Thom-Levin theorem is for vector field of manifold and is interested in equivalence and not conjugacy as here. Nevertheless the proof of this theorem is an adaptation of the one written by Golubitsky-Guillemin [GG73] P123-127.

Remark 4.2.2. The above theorem remains true if C is possibly non-compact but closed in L. For such a generalization, the proof bellow works as well.

Before proving Theorem 4.3, we need a few lemmas.

Lemma 4.2.3. *Let ξ be a $\mathcal{W}^\infty$-small vector field on $\mathcal{L} \times \mathbb{R}$ with zero $\mathbb{R}$-component. Then there is an automorphism H of $(L \times \mathbb{R}, \mathcal{L} \times \mathbb{R})$, which is a deformation of id_L satisfying:*

$$TH \circ \frac{\partial}{\partial t} \circ H^{-1} = -\xi + \frac{\partial}{\partial t},$$

Moreover, H is $\mathcal{W}^\infty$-close to the identity.

Proof. See [GG73] Sublemma 3.4 p125. $\square$

Lemma 4.2.4. *By using the same notations as in the above lemma, we have:*

(i) $\xi = -T\pi \circ TH \circ \frac{\partial}{\partial t} \circ H^{-1}$,

(ii) $\xi = TH \circ T\pi \circ TH^{-1} \circ \frac{\partial}{\partial t}$,

where $\pi : L \times \mathbb{R} \to L$ is the canonical projection.

Proof. The first statement of this lemma is obvious since $T\pi \circ \frac{\partial}{\partial t} = 0$ and $T\pi \circ \xi = \xi$. Applying TH^{-1} to both sides of the equation of Lemma 4.2.3, we get:

$$TH^{-1} \circ \big(-\xi + \frac{\partial}{\partial t}\big) = \frac{\partial}{\partial t} \circ H^{-1}.$$

Applying $T\pi$ on both sides above, we have:

$$-T\pi \circ TH^{-1} \circ \xi + T\pi \circ TH^{-1} \circ \frac{\partial}{\partial t} = 0.$$

As the $\mathbb{R}$-component of ξ is 0, so is the $\mathbb{R}$ component of $TH^{-1} \circ \xi$. Thus:

$$TH^{-1} \circ \xi = T\pi \circ TH^{-1} \circ \frac{\partial}{\partial t}.$$

By applying TH on both sides of the above equation, we have (ii). $\square$

Proof of Theorem 4.3. It is sufficient to show the existence of a deformation H of the identity such that:

$$F' = H^{-1} \circ F \circ H,$$

is equal to the trivial deformation F_0 on $C \times B$. We notice that this holds if and only if the following vector field is 0 on $C \times B$:

$$\tau_{F'} := T\pi \circ TF' \circ \frac{\partial}{\partial t}. \tag{6}$$

By assumption, there exists a $\mathcal{W}^\infty$-small vector field ξ on $(L \times \mathbb{R}, \mathcal{L} \times \mathbb{R})$, whose $\mathbb{R}$ component is zero and such that:

$$\tau_F = TF \circ \xi - \xi \circ F, \quad \text{on } W.$$

Let us construct H and so F' such that $\tau_{F'}$ is zero on $C \times B$. By applying Lemmas 4.2.3 and 4.2.4, we get the existence of a deformation:

$$H:\ L \times \mathbb{R} \to L \times \mathbb{R}$$

so that

$$\xi = -T\pi \circ TH \circ \frac{\partial}{\partial t_k} \circ H^{-1}.$$

As H is $\mathcal{W}^\infty$-close to the identity, it sends $C \times B$ into W.

We recall that $F' := H^{-1} \circ F \circ H$. Let p be in $C \times B$, $q := H(p)$ and $r := F \circ H(p)$. Using the fact that F and H are deformations, we have that:

$$TF' \circ \frac{\partial}{\partial t}(p) = \frac{\partial}{\partial t} \circ F'(p) + T\pi \circ TH^{-1} \circ \frac{\partial}{\partial t}(r) + TH^{-1}\Big[T\pi \circ TF \circ \frac{\partial}{\partial t}(q) - TF \circ \xi(q)\Big]$$

One the other hand, by using statement (ii) of Lemma 4.2.4, we have:

$$T\pi \circ TH^{-1} \circ \frac{\partial}{\partial t}(r) = TH^{-1} \circ \xi(r).$$

The two last equations imply that

$$TF' \circ \frac{\partial}{\partial t}(p) = \frac{\partial}{\partial t} \circ F'(q) + TH^{-1}\Big[\xi(r) + T\pi \circ TF \circ \frac{\partial}{\partial t}(q) - TF \circ \xi(q)\Big]$$

By assumption:

$$\tau_F = TF \circ \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \circ F = TF \circ \xi - \xi \circ F, \quad \text{on } W$$

We notice that $\tau_F = T\pi \circ TF \circ \frac{\partial}{\partial t}$. Thus, we have:

$$\tau_{F'}(p) := T\pi \circ TF' \circ \frac{\partial}{\partial t}(p) = T\pi \circ TH^{-1}\Big[-\tau_F(q) + T\pi \circ TF \circ \frac{\partial}{\partial t}(q)\Big] = 0$$

□

4.2.2 Proof of $\mathcal{I} \Rightarrow \mathcal{D}$ using the Thom-Levine theorem

Let f, C and $(L, \mathcal{L})$ be as stated in the theorem. In particular f is infinitesimally stable. We want to prove that f is stable by deformation, relatively to C. Let B be a bounded ball centered at 0 and W a neighborhood of $C \times B$ in L. By the adaptation of the Thom-Levin theorem, it is sufficient to show, for any CO^∞-small deformation $F:\ L \times \mathbb{R} \to L \times \mathbb{R}$ of f, the existence of a $\mathcal{W}^\infty$-small vector field $\xi \in \chi^\infty(\mathcal{L} \times \mathbb{R})^0$ with $\mathbb{R}$-component equal to 0 such that on W:

$$\tau_F := TF \circ \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \circ F = TF \circ \xi - \xi \circ F,$$

with $\partial/\partial t \in \chi^\infty(L \times \mathbb{R})$ the canonical unit vector field associated to $\mathbb{R}$.

Local version We first prove the existence of ξ locally:

Proposition 4.4. *Let $\hat{P}_1$ be a small plaque of $\mathcal{L}$ such that $\hat{P}_{i+1} := f^{-i}(\hat{P}_1)$ is disjoint from $\hat{P}_1$ for every $i > 0$. Let P_1 be a precompact plaque whose closure is included in $\hat{P}_1$. Let $P_{i+1} := f^{-i}(P_1)$, for every $i \geq 0$.*

Let $s > 0$ be such that $\hat{P}_s$ is non-empty. Then $(\hat{P}_i)_{i=1}^s$ consists of manifolds. Moreover, there exists a family of disjoint open neighborhoods $(U_i)_{i=1}^s$ of $(P_i)_{i=1}^s$ s.t. f sends $\Omega' := \coprod_{i=2}^s U_i$ into $\Omega := \coprod_{i=1}^s U_i$, and s.t. for any deformation F CO^∞-close to F_0, there exists $\xi \in \chi^\infty(\mathcal{L} \times \mathbb{R})^0$ which is $\mathcal{W}^\infty$-close to zero with $\mathbb{R}$-component equal to 0 and such that :

$$\tau_F(x,t) = TF \circ \xi(x,t) - \xi \circ F(x,t), \quad \forall x \in \Omega',\ t \in \mathbb{B}.$$

Proof. The fact that $(\hat{P}_i)_i$ is a family of manifolds follows from the transverse bijectivity of f.

The main interest of the proof is the usefulness of the algebraic tools developed by Mather ([Mat68a], [Mat69], [Mat68b]) and very well written by Tougeron [Tou95].

Let R'_i be the ring $C^\infty(\hat{P}_i)$ of smooth real functions on $\hat{P}_i$. For $t \in \mathbb{R}$, let R_i^t be the ring $C^\infty_{\hat{P}_i \times \{t\}}(\hat{P}_i \times \mathbb{R})$ of smooth germs at $\hat{P}_i \times \{t\}$ of smooth real functions on $\hat{P}_i \times \mathbb{R}$. We notice that R'_i is isomorphic to the quotient R_i^t/I_i^t, where I_i^t is the ideal of R_i^t formed by the germs equal to 0 on $\hat{P}_i \times \{t\}$. For $i \in \{1, \cdots, s-1\}$, we notice that the map

$$\phi_i^t :\ R_i^t \to R_{i+1}^t$$

$$\rho \mapsto \rho \circ F_{0|\hat{P}_{i+1} \times \mathbb{R}}$$

is a ring morphism that satisfies:

$$\phi_i^t(I_i^t) \subset I_{i+1}^t.$$

Also, the morphism ϕ_i^t induces on the quotient $R'_i \cong R_i^t/I_i^t \to R'_{i+1} \cong R_{i+1}^t/I_{i+1}^t$ the following ring morphism:

$$\phi'_i :\ R'_i \to R'_{i+1}$$

$$\rho \mapsto \rho \circ f.$$

Let $\tilde{R}_i$ be the ring $C^\infty(\hat{P}_i \times \mathbb{R})$ formed by the smooth real functions on $\hat{P}_i \times \mathbb{R}$.

Let X be in the space of smooth deformations $F :\ \coprod_{i=2}^s \hat{P}_i \times \mathbb{R} \to \coprod_{i=1}^s \hat{P}_i \times \mathbb{R}$ of $f_{|\coprod_{i=2}^s \hat{P}_i}$ endowed with the Whitney topology.

Let R_i^X be the ring $C^0_{F_0}(X, C^\infty(\hat{P}_i \times \mathbb{R}))$ formed by the germs at $F_{0|\coprod_{i=2}^s \hat{P}_i \times \mathbb{R}}$ of continuous maps from X into the space of smooth real functions on $\hat{P}_i \times \mathbb{R}$. We notice that $\tilde{R}_i$ is isomorphic to the quotient R_i^X/I_i^X where I_i^X denotes the ideal formed by the germs that vanish at $F_{0|\coprod_{i=2}^s \hat{P}_i \times \mathbb{R}}$. For $i < s$, we notice that the map

$$\phi_i^X :\ R_i^X \to R_{i+1}^X$$

$$(\rho_F)_{F \in X} \mapsto (\rho_F \circ F_{|\hat{P}_{i+1} \times \mathbb{R}})_{F \in X}$$

is a ring homomorphism that satisfies

$$\phi_i^X(I_i^X) \subset I_{i+1}^X.$$

Also, the homomorphism ϕ_i^X induces on the quotient $\tilde{R}_i \cong R_i^X/I_i^X \to \tilde{R}_{i+1} \cong R_{i+1}^X/I_{i+1}^X$ the following ring homomorphism:

$$\tilde{\phi}_i : \tilde{R}_i \to \tilde{R}_{i+1}$$

$$\rho \mapsto \rho \circ F_{0|\hat{P}_{i+1}\times\mathbb{R}}.$$

Let us give the formulation of the problem in these algebraic settings. For $i \in \{1, \dots, s\}$, we denote by:

- N_i' the R_i'-module $\chi^\infty(\hat{P}_i)$ of smooth vector fields on $\hat{P}_i$.
- N_i^t the R_i^t-module $\chi^\infty_{\hat{P}_i\times\{t\}}(\hat{P}_i\times\mathbb{R})^0$ of smooth germs of vector fields on $\hat{P}_i\times\mathbb{R}$ at $\hat{P}_i\times\{t\}$ with $\mathbb{R}$-component equal to zero.
- $\tilde{N}_i$ the $\tilde{R}_i$-module $\chi^\infty(\hat{P}_i \times \mathbb{R})^0$ of smooth vector fields on $\hat{P}_i \times \mathbb{R}$ with $\mathbb{R}$-component equal to zero.
- N_i^X the R_i^X-module $C^0_{F_0}(X, \chi^\infty(\hat{P}_i \times \mathbb{R})^0)$ of germs at $F_{0|\coprod_{i=2}^s \hat{P}_i\times\mathbb{R}}$ of continuous functions from X into $\chi^\infty(\hat{P}_i \times \mathbb{R})^0$.

For $i \in \{2, \dots, s\}$, let us denote by:

- M_i' the R_i'-module $\chi^\infty(f_{|\hat{P}_i})$ of smooth vector fields along $f_{|\hat{P}_i}$ (*i.e.* of smooth sections of $f^*_{|\hat{P}_i}T\hat{P}_{i-1}$),
- M_i^t be the R_i^t-modules $\chi^\infty_{\hat{P}_i\times\{t\}}(F_{0|\hat{P}_i\times\mathbb{R}})^0$ of germs at $\hat{P}_i \times \{t\}$ of vector fields along $F_{0|\hat{P}_i\times\mathbb{R}}$ with $\mathbb{R}$-component equal to zero.
- $\tilde{M}_i$ be the $\tilde{R}_i$-module $\chi^\infty(F_{0|\hat{P}_i\times\mathbb{R}})^0$ of smooth vector fields along $F_{0|\hat{P}_i\times\mathbb{R}}$ with $\mathbb{R}$-component equal to zero.
- M_i^X be the R_i^X-module of germs at x_0 of continuous sections σ of the trivial bundle $X \times \chi^\infty(\hat{P}_i \times \hat{P}_{i-1} \times \mathbb{R}) \to X$, such that $\sigma(F)$ belongs to $\chi^\infty(F_{|\hat{P}_i\times\mathbb{R}})^0$.

For $i \in \{2, \dots, s\}$, the following maps are homomorphisms of respectively R_i', R_i^t, $\tilde{R}_i$ and R_i^X-modules:

$$\alpha'_{i,i} : N_i' \to M_i' \qquad \alpha^t_{i,i} : N_i^t \to M_i^t$$
$$\xi \mapsto Tf \circ \xi \qquad \xi \mapsto TF^0 \circ \xi$$

$$\tilde{\alpha}_{i,i} : \tilde{N}_i \to \tilde{M}_i \qquad \alpha^X_{i,i} : N_i^X \to M_i^X$$
$$\xi \mapsto TF^0 \circ \xi \qquad (\xi_F)_F \mapsto (TF \circ \xi_F)_{F\in X}$$

Also for $i \in \{1, \ldots, s-1\}$, the following maps are module homomorphisms over ϕ'_i, ϕ^t_i, $\tilde{\phi}_i$ and ϕ^X_i respectively:

$$\begin{array}{cc} \alpha'_{i,i+1} : N'_i \to M'_{i+1} & \alpha^t_{i,i+1} : N^t_i \to M^t_{i+1} \\ \xi \mapsto \xi \circ f_{|\hat{P}_{i+1}} & \xi \mapsto \xi \circ F^0_{|\hat{P}_{i+1} \times \mathbb{R}} \end{array}$$

$$\begin{array}{cc} \tilde{\alpha}_{i,i+1} : \tilde{N}_i \to \tilde{M}_{i+1} & \alpha^X_{i,i+1} : N^X_i \to M^X_{i+1} \\ \xi \mapsto \xi \circ F^0_{|\hat{P}_{i+1} \times \mathbb{R}} & (\xi_F)_F \mapsto (\xi_F \circ F_{|\hat{P}_{i+1}})_{F \in X} \end{array}$$

For $\delta \in \{', t, \tilde{}, X\}$, we denote by

$$M^\delta := \bigoplus_{i=2}^{s} M^\delta_i \qquad \text{and} \qquad N^\delta := \bigoplus_{i=1}^{s} N^\delta_i$$

the Abelian sum of the modules $(M^\delta_i)^s_{i=2}$ and $(N^\delta_i)^s_{i=1}$ respectively. The modules M^δ and N^δ are modules over the rings $A := \bigoplus_{i=2}^s R^\delta_i$ and $B := \bigoplus_{i=1}^s R^\delta_i$ respectively. Let $I \cdot M^\delta$ (resp. $I \cdot N^\delta$) be the submodule of M^δ (resp. N^δ) spanned by elements of the form $i_k \cdot x_k$, with $i_k \in I^\delta_k$, $x_k \in M^\delta_k$ (resp. $x_k \in N^\delta_k$) and $k \in \{2, \ldots, s\}$ (resp. $k \in \{1, \ldots, s\}$). Let us consider the (additive) group morphism:

$$\alpha^\delta : N^\delta \to M^\delta$$

$$(\xi_i)^s_{i=1} \mapsto \left(\alpha^\delta_{i,i}(\xi_i) - \alpha^\delta_{i-1,i}(\xi_{i-1})\right)^s_{i=2}.$$

Let us show that the infinitesimal hypothesis implies the surjectivity of the map α': for every $\zeta \in \chi^\infty(f_{|\coprod_{i=2}^s \hat{P}_i})$, we can find a smooth function $r : \coprod_{i=1}^s \hat{P}_i \to (0, +\infty)$, which is f-invariant ($\forall x \in \coprod_{i=2}^s \hat{P}_i$, $r \circ f(x) = r(x)$) and such that $r \cdot \zeta$ can be extended to a smooth section ζ' of $f^*T\mathcal{L}$. Let ξ' be a vector field on $\mathcal{L}$ such that:

$$\xi' \circ f - Tf \circ \xi' = \zeta'.$$

Let $\xi'' := -\frac{1}{r} \cdot \xi'_{|\coprod_{i=1}^s \hat{P}_i}$. We notice that ξ'' is a smooth vector field on $\coprod_{i=1}^s \hat{P}_i$. Also:

$$Tf \circ \xi'' - \xi'' \circ f = \frac{1}{r}(\xi' \circ f - Tf \circ \xi') = \frac{\zeta'}{r} = \zeta, \quad \text{on } \coprod_{i=2}^s \hat{P}_i.$$

Let us show that the proposition is proved if we show that α^X sends $I^X \cdot N^X$ onto $I^X \cdot M^X$.

For F in X, let $\tau_F := T\pi \circ TF \circ \frac{\partial}{\partial t}$. We notice that $(\tau_F)_{F \in X}$ is an element of $I^X \cdot M^X$. Thus, we have the existence of $(\xi_F)_{F \in X} \in I^X \cdot N^X$ and of a neighborhood X' of $F_{0|\coprod_{i=2}^s \hat{P}_i}$ in X such that for every $F \in X'$, we have on $\coprod_{i=2}^s \hat{P}_i$:

$$\tau_F := TF \circ \xi_F - \xi_F \circ F.$$

Let us construct $(U_i)_i$. Let $\hat{T}$ be a locally compact metric space such that a small neighborhood $\hat{U}_1$ of $\hat{P}_1$ is isomorphic to the product $\hat{P}_1 \times \hat{T}$. Let $\tau_0 \in \hat{T}$ s.t. this isomorphism

sends $\hat{P}_1$ to $\hat{P}_1 \times \{\tau_0\}$. Since f is transversally bijective, for $\hat{U}_1$ small enough, we notice that a neighborhood $\hat{U}_i$ of $\hat{P}_i$ is canonically isomorphic to $\hat{P}_i \times \hat{T}$, and this isomorphism sends $\hat{P}_i$ onto $\hat{P}_i \times \{\tau_0\}$. For $\hat{T}$ sufficiently small the open subsets $(\hat{U}_i)_i$ are disjoint. Let $\hat{\Omega} := \coprod_{i=1}^s \hat{U}_i$ and $\hat{\Omega}' := \coprod_{i=2}^s \hat{U}_i$. Let $F_0 : L \times \mathbb{R} \to L \times \mathbb{R}$ be the trivial deformation of f.

Let exp be the exponential map associated to a complete metric on $\coprod_{i=1}^s \hat{P}_i$. Let $r \in C^\infty(\coprod_{i=1}^s \hat{P}_i \times \mathbb{R}, \mathbb{R})$ be a compactly supported function equal to 1 on a neighborhood of $\coprod_{i=1}^s P_i \times B$. There exists a CO^∞-neighborhood V_F of the trivial deformations F^0 of f and a neighborhood T of $\tau_0 \in \hat{T}$ such that for every deformation $F \in V_F$ and $\tau \in T$ the following map is well defined:

$$F_\tau : \coprod_{i=2}^s \hat{P}_i \times \mathbb{R} \to \coprod_{i=1}^s \hat{P}_i \times \mathbb{R}$$

$$(x,t) \mapsto \left[\begin{array}{ll} \left(\exp_{f(x,\tau_0)}\left(r(x,t) \cdot \exp^{-1}_{f(x,\tau_0)}\left(F(x,\tau,t)\right)\right), t\right) & \text{if } r(x,t) \neq 0 \\ \left(f(x,\tau_0), t\right) & \text{else} \end{array} \right.$$

with in particular the restriction of F to $\coprod_{i=2}^s P_i \times \{\tau\} \times \mathbb{R}$ canonically identitified to a map from $\coprod_{i=2}^s P_i \times \mathbb{R}$ into $\coprod_{i=1}^s \hat{P}_i \times \mathbb{R}$. Also, when F is CO^∞-close to F_0 and τ is close to τ_0, then F_τ is $\mathcal{W}^\infty$-close to $F_{0|\coprod_{i=2}^s \hat{P}_i \times \mathbb{R}}$. We suppose V_F and T sufficiently small such that F_τ belongs to X' and such that $r \circ F_\tau$ is equal to 1 on $\coprod_{i=2}^s P_i \times B$, for every $F \in V_F$ and $\tau \in T$.

Let $\rho \in C^0(\hat{T}, \mathbb{R})$ be a function equal to 1 on a neighborhood T' of $\tau^0 \in T$ and to 0 off T. For $F \in V_F$, let $U_i := P_i \times T'$, $\Omega := \coprod_{i=1}^s U_i$, $\Omega' := \coprod_{i=2}^s U_i$ and:

$$\xi := z \in L \times \mathbb{R} \mapsto \begin{cases} \rho(\tau) \cdot r(x,t) \cdot \xi_{F^\tau}(x,t) & \text{if } z = (x,\tau,t) \in \coprod_{i=1}^s \hat{P}_i \times T \times \mathbb{R} \\ 0 & \text{else} \end{cases}.$$

We notice that ξ belongs to $\chi^\infty(\mathcal{L} \times \mathbb{R})^0$ and that we have on Ω':

$$\tau_F := T\pi \circ TF \circ \frac{\partial}{\partial t} = TF \circ \xi - \xi \circ F.$$

Also when F is CO^∞-close to F_0, then ξ is $\mathcal{W}^\infty$-small. Hence the proposition is shown.

The proof that the surjectivity of all $(\alpha^t)_{t \in \mathbb{R}}$, implies the surjectivity of $\tilde{\alpha}$ is easy. It will be done at the end.

To show that the surjectivity of α' implies the one of α^t, and that the one of $\tilde{\alpha}$ implies the one of α^X and that $\alpha^X(I^X \cdot N^X) = I^X \cdot M^X$, we shall use the following techniques of Mather.

The algebraic Machinery Let R and S be rings with units. Let I and J be *Jacobson ideals* (this means that for every $z \in J$, the element $1+z$ is invertible) in R and S respectively. Let $\phi : R \to S$ be a ring homomorphism which sends I into J.

Definition 4.2.5. The homomorphism $\phi : (R, I) \to (S, J)$ is *adequate* if the following condition is satisfied: Let A be a finitely generated R-module. Let B and C be S-modules, with C finitely generated over S. Let $\beta : B \to C$ be a homomorphism of S-modules. Let

$\alpha : A \to C$ be a homomorphism over ϕ (*i.e.* $\alpha(a+b) = \alpha(a) + \alpha(b)$ and $\alpha(r \cdot a) = \phi(r) \cdot \alpha(a)$, for $a, b \in A$ and $r \in R$). Suppose that:

$$\alpha(A) + \beta(B) + J \cdot C = C.$$

Then we can conclude

$$\alpha(A) + \beta(B) = C \qquad \text{and} \qquad \alpha(I \cdot A) + \beta(J \cdot B) = J \cdot C.$$

Let us illustrate the above definition by the following non-trivial examples shown by Mather in [Mat68a]-[Mat69]-[Mat68b] and rewritten in this form by Tougeron [Tou95]:

Theorem 4.5. *For any $i \in \{2, \dots, s\}$, the ring homomorphisms:*

$$\phi_i^t : (R_i^t, I_i^t) \to (R_{i+1}^t, I_{i+1}^t)$$

and

$$\phi_i^X : (R_i^X, I_i^X) \to (R_{i+1}^X, I_{i+1}^X)$$

are adequate.

This is the algebraic theorem of Mather:

Theorem 4.6 (Mather)**.** *Let (R_i, I_i), $i = 1, \dots, s$ be rings with units where I_i is a Jacobson ideal for every i. Let*

$$(R_1, I_1) \stackrel{\phi_1}{\to} (R_2, I_2) \stackrel{\phi_2}{\to} \cdots \stackrel{\phi_{s-1}}{\to} (R_s, I_s)$$

be a sequence of adequate homomorphisms. For every i, let N_i and M_i be R_i-modules, finitely generated (with possible exception for N_s). Put for $i < j$:

$$\phi_{ij} = \phi_j \circ \cdots \circ \phi_i$$

and ϕ_{ii} the identity of R_i. For $j \geq i$, let $\alpha_{ij} : N_i \to M_j$ be a module-homomorphism over ϕ_{ij}. Let $N := \oplus_{i=1}^s N_i$ and $M := \oplus_{i=1}^s M_i$ as the direct sums of Abelian groups and let

$$\alpha : N \to M$$

be given by

$$\alpha(\xi_1, \dots, \xi_s) = (\alpha_{11}(\xi_1), \alpha_{12}(\xi_1) + \alpha_{22}(\xi_2), \dots, \alpha_{1s}(\xi_1) + \cdots + \alpha_{ss}(\xi_s)).$$

Suppose that

(I) $\qquad \alpha(N) + \sum_{i=1}^s I_i M_i = M.$

Then:

(II) $\qquad \alpha$ *sends N onto M.*

(III) *Moreover α sends $\sum_{i=1}^s I_i \cdot N_i$ onto $\sum_{i=1}^s I_i \cdot M_i$.*

Remark 4.2.6. We can illustrate the morphism α by the following diagram:

$$\begin{array}{ccccccccc}
(R_1, I_1) & \stackrel{\phi_1}{\rightarrow} & (R_2, I_2) & \stackrel{\phi_2}{\rightarrow} & \cdots & \stackrel{\phi_{n-1}}{\rightarrow} & (R_s, I_s) \\
N_1 & \oplus & N_2 & \oplus & \cdots & \oplus & N_s \\
\alpha_{11} \downarrow & \alpha_{12} \searrow & \alpha_{22} \downarrow & & \searrow \cdots \searrow & \searrow & \downarrow \alpha_{ss} \\
M_1 & \oplus & M_2 & \oplus & \cdots & \oplus & M_s.
\end{array}$$

Remark 4.2.7. For $s = 1$ this theorem is the Nakayama's lemma. For $s = 3$ is was shown by F. Latour.

As the proof is purely algebraic, we will prove this theorem at the end of this work. Let us conclude the proof of the proposition.

If we omit the exponent $X, t, \ ' \ , \tilde{}$, and we put $M_1 = 0$ and $\alpha_{ji} = 0$ when $j < i - 1$, it follows from the last theorem that the surjectivity of α' implies the one of α^t, and that the surjectivity of $\tilde{\alpha}$ implies the one of α^X with $\alpha^X(I^X N^X) = I^X M^X$.

Thus it only remains to prove that the surjectivity of all $(\alpha^t)_{t\in\mathbb{R}}$ implies the one of α'. Let $\tau \in \chi^\infty(F_0)^0$. For $t \in \mathbb{R}$, as α^t is surjective, there exists a germ $\xi_t \in A_t$ such that $\tau_t := \alpha_t(\xi_t)$. The germs ξ_t is defined on a neighborhood V_t of $\coprod_{i=1}^s \hat{P}_i \times \{t\}$ in $\coprod_{i=1}^s \hat{P}_i \times \mathbb{R}$. By shrinking a slice $\hat{\Omega}$ and then by shrinking V_t for every $t \in \mathbb{R}$, we may suppose that V_t is of the form $(\coprod_{i=1}^s \hat{P}_i) \times W_t$, where W_t is a neighborhood of $t \in \mathbb{R}$. Let $(W_j)_{j\in\mathbb{N}}$ be a locally finite subcovering of $(W_t)_{t\in\mathbb{R}}$. By locally finite we mean that there exists for every point $t \in \mathbb{R}$ a finite number of integers $j \in \mathbb{N}$ such that W_j intersects a neighborhood of t. By subcovering we mean that for every $j \geq 0$ there exists t_j s.t. W_j is included in W_{t_j}. Let $(\rho_j)_j$ be a partition of the unity subordinate to $(W_j)_j$. Let $\pi' : \coprod_{i=1}^s \hat{P}_i \times \mathbb{R} \to \mathbb{R}$ be the projection on the second coordinate. We notice that:

$$\pi' \circ F_0 = \pi' \quad \text{on} \ \coprod_{i=2}^s \hat{P}_i \times \mathbb{R},$$

since the map F_0 is a deformation of f. Let $\xi := \sum_{j=1}^\infty \rho_j \circ \pi' \cdot \xi_{t_i}$. We notice that ξ is sent by α' to τ. This concludes the proof of the proposition. □

Remark 4.2.8. We notice that one easily simplifies the above proof to show that under the hypotheses of Theorem 1.7 of Mather, for every deformation $(F_{ij})_{[i,j]\in A}$ which is $\mathcal{W}^\infty$-close to the trivial deformation of $(f_{ij})_{[i,j]\in A}$ there exists a $\mathcal{W}^\infty$-small vector vector field $\xi \in \chi^\infty(\coprod_{i\in V} M_i \times \mathbb{R})^0$ such that

$$(TF_{ij} \circ \xi - \xi \circ f_{ij})_{|M_i} = TF_{ij} \circ \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \circ F_{ij|M_i},$$

for every $[i,j]\in A$.

Thus by using Remark 4.2.2 with C the disjoint union of the manifold from which an arrow start, we get the proof of the Theorem 1.7.

4.3 From local to global: proof of Theorem 2.3

Let f be an endomorphism of $\mathcal{L}$ and let C be a compact subset of L as in the statement of the theorem. We notice that by compactness of C, there exists a compact neighborhood V of K s.t. for every $x\in V$ some iterate $f^n(x)$ does not belong to V, for $n\in\mathbb{N}$. Let $W := f(V)\setminus int(V)$. For every $p\in W$, we define $(p_i)_{i\geq 1}$ inductively: $P := \{p\}$; $P_{i+1} := f^{-1}(P_i)\cap V$. We notice that P_N is empty. Let s_p be maximal s.t. P_{s_p} is not empty. We define $(\hat{P}_i^p)_{i=1}^{s_p}$ inductively: $\hat{P}_1^p$ is a small plaque that contains p and $\hat{P}_{i+1}^p := f^{-1}(\hat{P}_1^p)$. We notice that for $\hat{P}_1^p$ sufficiently small, $\hat{P}_i^p$ is disjoint from $\hat{P}_1^p$ (else we would have a cycle in V). Let P_1^p be a precompact neighborhood of P_1^p and let $(U_i^p)_{i=1}^{s_p}$ be the open subsets provided by Proposition 4.4.

Also, by shrinking if necessarily, we may suppose that the closure of U_1^p does not intersect C. As W is compact, we can extract from $(U_1^p)_{p\in C}$ a finite subcover $(U_1^{p_j})_{j\geq 1}$ of W. As W is compact and $\cap_{k=0}^N f^{-n}(V)$ is empty, we notice that for every f' CO^∞-close enough to f, for every $x\in C$, there exists an integer $n\geq 1$, s.t.:

$$f^n(x)\in\Delta := \cup_{j\geq 1}\breve{U}_1^{p_j}.$$

For every CO^∞-small deformation F of f, for every j, there exists $\xi_j\in\chi^\infty(\cup_{i=1}^{s_j}U_i^{p_j}\times\mathbb{R})^0$, such that

$$\tau_F = TF\circ\xi_j - \xi_j\circ F,\quad \text{on } \cup_{i=2}^{s_j} U_i^{p_j}\times\mathbb{R}$$

Let $(\rho_1^j)_{j=1}^N$ be a partition of the unity subordinate to $(U_1^{p_j})_j$.

$$\text{Let } \rho_2^j = (x,t)\in L\times\mathbb{R}\mapsto \begin{cases} \rho_1^j\circ p_1\circ F^i(x,t) & (x,t)\in U_i^{p_j}\times\mathbb{R} \\ 0 & \text{else}\end{cases},$$

with $p_1:\ L\times\mathbb{R}\to L$ the canonical projection.

Let $R := \sum_{j=1}^N \rho_2^j$. Since $(U_i^{p_j})_{i,j}$ is finite, the function R is well defined and is a smooth morphism of $\mathcal{L}\times\mathbb{R}$ into $\mathbb{R}$. Let $\rho_j := \frac{\rho_2^j}{R}$ and $\xi := x\mapsto \sum_{j=1}^N \rho_j(x)\xi_j(x)$. As C is disjoint from the closure of $\cup_{j=1}^N U_1^{p_j}$ we have for all $(x,t)\in C\times\mathbb{R}$,

$$\tau_F(x,t) = TF\circ\xi(x,t) - \xi\circ F(x,t).$$

4.4 Infinitesimal stability for manifold implies infinitesimal stability for embedded lamination

We recall that to prove Theorem 2.3, we only used the surjectivity of the map:

$$\chi^\infty(\coprod_{i=1}^s P_i)\to\chi^\infty(f_{|\coprod_{i=2}^s P_i})$$

$$\sigma \mapsto \sigma \circ f - Tf \circ \sigma.$$

But this surjectivity is an easy consequence of the infinitesimal stability of f stated in hypothesis (ii): for every $\tau \in \chi^\infty(f_{|\coprod_{i=2}^s P_i})$, we can find a smooth function $r : \coprod_{i=1}^s P_i \to (0, +\infty)$, which is f-invariant ($\forall x \in \coprod_{i=2}^s P_i$, $r \circ f(x) = r(x)$) and such that $r \cdot \tau$ can be extended to a smooth section τ' of f^*TM. Let ξ' be a vector field on $M \setminus \hat{\Omega}$ such that on this domain:

$$\xi' \circ f - Tf \circ \xi' = \tau'$$

Let $\xi'' := \frac{1}{r} \cdot \xi'_{|\coprod_{i=1}^s P_i}$. We notice that ξ'' is a smooth vector field on $\coprod_{i=1}^s P_i$. Also

$$\xi'' \circ f - Tf \circ \xi'' = \frac{1}{r}(\xi' - Tf \circ \xi') = \frac{\tau'}{r} = \tau$$

Now we have to transform ξ'' to a vector field tangent to $\coprod_{i=1}^s P_i$. Let $N_1 \to P_1$ be the smooth vector bundle whose fiber at $x \in P_1$ is $T_x P_1^\perp$. Let $N_i \to P_i$ be the smooth vector bundle whose fiber at $x \in P_i$ is $T_x f^{-i+1}(T_{f^{i-1}(x)} P_1^\perp)$. By transversality of f to $\mathcal{L}$, P_1 is also a smooth vector bundle. Let π be the projection of $TM_{|\coprod_{i=1}^s P_i}$ onto $T(\coprod_{i=1}^s P_i)$ parallelly to $\coprod_i N_i \to \coprod_{i=1}^s P_i$. We notice that $\xi := \pi \circ \xi''$ satisfies the requested properties.

4.5 Proof of Mather's algebraic theorem

The proof of this theorem comes from an unpublished manuscript of Mather. It was then rewritten by Baas in a manuscript as well unpublished. Here we only copy the last proof. The first step is to show that it is sufficient to prove the theorem for $M_1 = 0$. Let us assume that this has been proved and from this prove the theorem. Put

$$M' = \bigoplus_{i=2}^s M_i$$

and

$$\alpha' : N \to M'$$

is given by

$$\alpha'(n_1, \ldots, n_s) = (\alpha_{12}(n_1) + \alpha_{22}(n_2), \ldots, \alpha_{1s}(n_1) + \cdots + \alpha_{ss}(n_s)).$$

The hypothesis (I) clearly implies:

(I') $\qquad \alpha'(N) + \sum_{i=2}^s I_i M_i = M'.$

Then the special case of the theorem with $M_1 = 0$ which we assume gives:

$$\alpha'(N) = M' \quad \text{and} \quad \alpha'\Big(\sum_{i=1}^s I_i \cdot N_i\Big) = \sum_{i=2}^s I_i \cdot M_i.$$

$$\text{Let} \quad N_1' = (\alpha_{12} + \cdots + \alpha_{1,s})^{-1} \alpha\Big(\sum_{i=2}^s N_i\Big).$$

It is now sufficient to prove

$$\alpha_{11}(N_1') = M_1.$$

This would give (I) and also imply:

$$\alpha_{11}(I_1 N_1') = I_1 M_1$$

which together with

$$(\alpha_{12} + \cdots + \alpha_{1s})(I_1 \cdot N_1') \subset \alpha\Big(\sum_{i=2}^{s} I_i \cdot N_i\Big)$$

gives (III). Let us prove that

$$\alpha_{11}(N_1') = M_1$$

follows from this implication of (III'):

$$M = \alpha(N) + \alpha'\Big(\sum_{i=1}^{s} I_i \cdot N_i\Big) + I_1 \cdot M_1 = \alpha\Big(\sum_{i=2}^{s} N_i\Big) + \alpha'(I_1 \cdot N_1) + \alpha(N_1) + I_1 \cdot M_1.$$

For $m_1 \in M_1$, there exist $n_1' \in I_1 \cdot N_1$, $m_1' \in I_1 \cdot M_1$ and $n_i \in N_i$, for $i \in \{1, \ldots, s\}$, such that:

$$m_1 \oplus 0 \oplus \cdots \oplus 0 = \alpha(n_2 + \cdots + n_s) + \alpha'(n_1') + \alpha(n_1) + (m_1' \oplus 0 \cdot \oplus 0).$$

Componentwise this gives:

$$b_1 = \alpha_{11}(n_1) + m_1' \tag{7}$$

$$0 = (\alpha_{1s} + \cdots + \alpha_{2s})(n_1) + (\alpha_{12} + \cdots + \alpha_{1s})(n_1') + \alpha(n_2 + \cdots + n_s). \tag{8}$$

From the second equation, it follows that the point $n_1 + n_1'$ belongs to N_1'. Hence:

$$m_1 = \alpha_{11}(n_1 + n_1') + (m_1' - \alpha_{11}(n_1')) \in \alpha_{11}(N_1') + I_1 M_1.$$

So we have shown

$$M_1 = \alpha_{11}(N_1') + I_1 M_1$$

and by applying Nakayama's Lemma, we get:

$$\alpha_{11}(N_1') = M_1.$$

And this finishes the proof of the theorem, assuming its holds for $M_1 = 0$. So the next step is to prove the theorem for $M_1 = 0$. This is done by induction on s. For $s = 1$, the theorem is trivial. So assume the theorem inductively for $s - 1$.
Put

$$N_1^* = N_1 \bigotimes_{R_1} R_2$$

where R_2 is regarded as an R_1-module via ϕ_1. Let

$$\alpha_{1j}^* := \alpha_{1j} \otimes \phi_{2j} : \; N_1^* \to M_j, \; j \geq 2$$

and

$$N^* = N_2 \oplus \cdots \oplus N_s \oplus N_1^*$$

$$M = M_2 \oplus \cdots \oplus M_s$$

Define

$$\alpha^* : \ N^* \to M$$

by

$$\alpha^*(n_1, \ldots, n_s) = \Big(\alpha_{22}(n_2) + \alpha_{12}^*(n_1), \ldots, \alpha_{1s}^*(n_1) + \cdots + \alpha_{ss}(n_s)\Big)$$

Clearly N_1^* is a finitely generated R_2-module and α_{1j}^* is a homomorphism over ϕ_{1j}. Now (I) implies

$$\alpha^*(N^*) + \sum_{i=2}^{s} I_i M_i = M.$$

And by the induction hypothesis we conclude

(II^*) α^* sends N^* onto M.

(III^*) Moreover α^* sends $\sum_{i=2}^{s} I_i \cdot N_i + I_2 N_1^*$ onto $\sum_{i=2}^{s} I_i \cdot M_i$.

Let

$$\beta = id \otimes \phi_1 : \ N_1 = N_1 \bigotimes_{R_1} R_1 \to N_1^* = N_1 \bigotimes_{R_1} R_2.$$

Then $\alpha_{1j}^* \circ \beta$ is equal to α_{1j}, for every j. . Let

$$C := (\alpha_{12}^* + \cdots + \alpha_{1s}^*)^{-1} \cdot \alpha\Big(\sum_{i=2}^{s} N_i\Big)$$

and this is an R_2-submodule of N_1^*. Let $n^* \in M_1^*$ and $m = \alpha^*(n^*) \in M$. Then by our assumption (I):

$$m = \alpha(n_1 + \cdots + n_s) + m_2' + \cdots + m_s',$$

where $n_i \in N_i$ and $m_i' \in I_i \cdot M_i$. By (III^*)

$$m_2' + \cdots + m_s' = \alpha^*(n_1^* + \cdots + n_s^*)$$

where $n_1^* \in I_2 \cdot N_1^*$, and $n_i^* \in I_i N_i$, for $i \geq 2$. Hence

$$m = \alpha\Big((n_1 + (n_2 + n_2^*) + \cdots + (n_s + n_s^*)\Big) + \alpha^*(n_1^*).$$

Therefore

$$\alpha^*(n^* - \beta(n_1) - n_1^*) = m - \alpha(n_1) - \alpha^*(n_1^*) = \alpha\big((n_2 + n_2^*) + \cdots + (n_s + n_s^*)).$$

Put

$$c = n^* - \beta(n_1) - n_1^* \in C$$

Since

$$n^* = c + \beta(n_1) + n_1^* \in C + \beta(N_1) + I_2 N_1^*$$

We get

$$N_1^* = C + \beta(N_1) + I_2 N_1^*.$$

Now since ϕ_2 is adequate we deduce

$$(II^{**}) \qquad N_1^* = C + \beta(N_1).$$

$$(III^{**}) \qquad I_2 N_1^* = I_2 C + \beta(I_1 N_1).$$

Clearly (II^*) and (II^{**}) give (II) and (III^*). Also (III^*) and (III^{**}) give (III). This finishes the proof of the theorem.

□

4.6 Proof of Lemma 1.3.10

Let $\xi \in \chi^\infty(f_{|M'})$. By (c) there exists $\sigma_0 \in \chi(M')$ such that:

$$(\sigma_0 \circ f - Tf \circ \sigma_0)_{|U} = \xi_{|U}.$$

By restricting a slice U s.t. (a) and (b) are still satisfied, we may suppose that σ_0 can be smoothly extended to M.

We define inductively $(\sigma_n)_{n\geq 0} \in \prod_n \chi(f_{|\cup_{k=0}^n f^k(U)})$ by:

$$\sigma_{n+1} : x \mapsto \begin{cases} \sigma_n(x) & \text{if } x \in \cup_{k=0}^n f^k(U) \\ \xi \circ f^{-1}(x) + Tf \circ \sigma_n \circ f^{-1}(x) & \text{if } x \in f^{n+1}(U) \setminus U \end{cases}$$

We notice that $(\sigma_n)_n$ is well defined and is locally eventually constant on the open subset $U^+ := \cup_{k\geq 0} f^k(U)$. Thus $(\sigma_n)_n$ converges to some section $\sigma_0^+ \in \chi(U^+)$. Also σ^+ satisfies:

$$(\sigma_0^+ \circ f - Tf \circ \sigma_0^+)_{|U^+} = \xi_{|U^+}.$$

Let us define $(\sigma_n^+)_n \in \prod_{n\geq 0} \chi(f_{|f^{-n}(U^+)})$ by induction. Let $n \geq 0$ and suppose σ_n^+ constructed. For this end we notice that the forward orbit $O^+(\Sigma)$ of the singularities of f is closed and that on $M'' := M' \setminus O^+(\Sigma)$, the map $S := x \mapsto \ker(T_x f)$ is a smooth section of the Grassmannian of TM''. Let p be the orthogonal projection of TM'' onto S. Remember that σ_0 can be smoothly extended to M'. This implies that σ_n can be smoothly extended to M'. Let $\sigma_n^s \in \chi(M')$ be a smooth extension of $p \circ \sigma^+_{n|M''}$ such that $\sigma_n^s(x)$ belongs to $S(x)$, for every $x \in M'$. We can now define inductively for $n \geq 0$:

$$\sigma_{n+1}^+ := x \in f^{-n-1}(U^+) \mapsto \begin{cases} \sigma_n^+(x) & \text{if } x \in f^{-n}(U^+) \\ (Tf^{-1}_{|S(x)^\perp}\big(\sigma_n^+ \circ f(x) - \xi(x)\big) & \text{if } x \in f^{-n-1}(x) \end{cases}$$

We notice that $(\sigma_n^+)_n$ is well defined and eventually constant on $\hat{U} := \cup_{n\geq 0} f^{-n}(U^+)$. Thus $(\sigma_n^+)_n$ converges to some section $\hat{\sigma} \in \chi(\hat{U})$. Also $\hat{\sigma}$ satisfies:

$$(\hat{\sigma} \circ f - Tf \circ \hat{\sigma})_{|\hat{U}} = \xi_{|\hat{U}}.$$

Let U' be an open neighborhood of the singularities satisfying Property (b) of the lemma, such that U contains the closure of U'.

Let $\hat{U}' := \cup_{n\geq 0} f^{-n}(\cup_{k\geq 0} f^k(U'))$ and let $\check{U}$ be the complement of $\hat{U}'$ in M'. We notice that $(\hat{U}, \check{U})$ is an open cover of M'. Thus to finish the proof of the lemma, it is sufficient to find a partition of the unity $(r, 1-r)$ subordinate to this cover, which is f-invariant ($r \circ f = r$) and to find a section $\check{\sigma} \in \chi(\check{U})$ such that

$$(\check{\sigma} \circ f - Tf \circ \check{\sigma})_{|\check{U}} = \xi_{|\check{U}}.$$

Then we notice that $\sigma := r \cdot \hat{\sigma} + (1-r) \cdot \check{\sigma}$ satisfies the requested property.

Let V be a manifold with boundary such that A is the maximal invariant of V (*i.e* $A = \cap_n f^n(V)$) and f sends V into its interior.

Let us construct r. As we are here in the diffeomorphism case, the construction a partition of the unity $(r_1, 1-r_1)$ subordinate to the cover $(V \cap \hat{U}, V \cap \check{U})$ and f-invariant is classic. Then we define $r := x \in M' \mapsto r \circ f^n(x)$, if $x \in f^{-n}(V)$ which is convenient for our purpose.

Let us construct $\check{\sigma}$. Let $D := (V \setminus f(V)) \cap \check{U}$. Let $\partial^+ D := \partial D \cap f(V) \cap \check{U}$ and $\partial^- D = \partial D \cap int(V)^c \cap \check{U}$, with ∂D the boundary of D. On a neighborhood of $\partial^- D$ we define $\check{\sigma}_+ = 0$ and on a neighborhood of $\partial^- D$ we define $\check{\sigma}_- = Tf^{-1} \circ (\check{\sigma}_+ \circ f - \xi)$. Then we chose a section $\check{\sigma}_0$ of $\chi^\infty(M)$ equal to $\check{\sigma}_+ = 0$ on a neighborhood of $\partial^+ D$ and to $\check{\sigma}_-$ on a neighborhood of $\partial^- D$.

As for the construction of $\hat{\sigma}$, we define then $\check{\sigma}$ on $\check{U}^+ := \cup_{n\geq 0} f^n(D)$ and finally on $\check{U} = \cup_{n\geq 0} f^n(\check{U}^+)$.

□

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